1. Suppose that \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) is defined by

\[
\begin{align*}
f(x, y, z) &= (x^2 + yz, \sin(xyz) + z).
\end{align*}
\]

(a) Why is \( f \) differentiable on \( \mathbb{R}^3 \)? Compute the Jacobian matrix of \( f \) at \((x, y, z) = (-1, 0, 1)\).

(b) Are there any directions in which the directional derivative of \( f \) at \((-1, 0, 1)\) is zero? If so, find them.

Solution.

- (a) The partial derivatives of the component functions of \( f \) exist and are continuous on \( \mathbb{R}^3 \), so \( f \) is differentiable on \( \mathbb{R}^3 \).

- Explicitly, we write \( f = (f_1, f_2) \) where \( f_1, f_2 : \mathbb{R}^3 \to \mathbb{R} \) are given by

\[
\begin{align*}
f_1(x, y, z) &= x^2 + yz, \\
f_2(x, y, z) &= \sin(xyz) + z.
\end{align*}
\]

The partial derivatives are

\[
\begin{align*}
\frac{\partial f_1}{\partial x}(x, y, z) &= 2x, \\
\frac{\partial f_1}{\partial y}(x, y, z) &= z, \\
\frac{\partial f_1}{\partial z}(x, y, z) &= y, \\
\frac{\partial f_2}{\partial x}(x, y, z) &= yz \cos(xyz), \\
\frac{\partial f_2}{\partial y}(x, y, z) &= xz \cos(xyz), \\
\frac{\partial f_3}{\partial z}(x, y, z) &= xy \cos(xyz) + 1,
\end{align*}
\]

and the Jacobian matrix of \( f \) is

\[
[df] = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}
\end{pmatrix}
\]

- In particular,

\[
\begin{align*}
\frac{\partial f_1}{\partial x}(-1, 0, 1) &= -2, & \frac{\partial f_1}{\partial y}(-1, 0, 1) &= 1, & \frac{\partial f_1}{\partial z}(-1, 0, 1) &= 0, \\
\frac{\partial f_2}{\partial x}(-1, 0, 1) &= 0, & \frac{\partial f_2}{\partial y}(-1, 0, 1) &= -1, & \frac{\partial f_3}{\partial z}(-1, 0, 1) &= 1.
\end{align*}
\]
Therefore, the Jacobian matrix of \( f \) at \((-1,0,1)\) is

\[
[df(-1,0,0)] = \begin{pmatrix}
-2 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

- (b) The directional derivative of \( f \) at \((-1,0,1)\) in the direction \( h \) is

\[
D_h f(-1,0,1) = df(-1,0,1)h.
\]

If \( h = (a,b,c) \), then the directional derivative has components

\[
[D_h f(-1,0,1)] = \begin{pmatrix}
-2 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = \begin{pmatrix}
-2a + b \\
-b + c
\end{pmatrix}
\]

- This is zero if \( b = 2a \) and \( c = b \), or \( h = k(1,2,2) \) where \( k \) is an arbitrary constant. Thus, normalizing \( h \) to unit vector, with a convenient choice of sign, the directional derivative of \( f \) at \((-1,0,1)\) is zero in the direction

\[
h = \frac{1}{3}(1,2,2).
\]
2. Suppose that $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$f(t) = (t, t^2, t^3), \quad g(x, y, z) = x^2e^{yz},$$

and $h = g \circ f : \mathbb{R} \to \mathbb{R}$ is their composition.

(a) Use the chain rule to compute $h'(1)$.
(b) Find $h(t)$ and compute $h'(1)$ directly.

Solution.

- (a) Note that $f$ is differentiable on $\mathbb{R}$, since each of its component functions is differentiable, and $g$ is differentiable on $\mathbb{R}^3$ since its partial derivatives exist and are continuous.

- The Jacobian matrix of $f$ at $t = 1$ is

$$[df(1)] = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} \bigg|_{t=1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

- We have $f(1) = (1, 1, 1)$, and the Jacobian matrix of $g$ at $(1, 1, 1)$ is

$$[dg(f(1))] = \left( \begin{array}{ccc} 2xe^{yz} & x^2ze^{yz} & x^2ye^{yz} \end{array} \right)_{(x,y,z)=(1,1,1)} = \left( \begin{array}{ccc} 2e & e & e \end{array} \right).$$

- By the chain rule, $h = g \circ f$ is differentiable on $\mathbb{R}$ and

$$h'(1) = [dg(f(1))][df(1)] = \left( \begin{array}{ccc} 2e & e & e \end{array} \right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2e + 2e + 3e = 7e.$$

- We have

$$h(t) = g(t, t^2, t^3) = t^2e^{t^5}.$$  

Thus, by the product and chain rules from one-variable calculus,

$$h'(1) = \left( 2t \cdot e^{t^5} + t^2 \cdot 5t^4 e^{t^5} \right) \bigg|_{t=1} = 7e.$$
3. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x, y) = \begin{cases} 
    x^{4/3} \sin(y/x) & \text{if } x \neq 0, \\
    0 & \text{if } x = 0.
\end{cases}
\]

Where is \( f \) is differentiable?

**Solution.**

- The function \( f \) is differentiable at every point of \( \mathbb{R}^2 \).
- By the chain and product rules, the partial derivatives of \( f \),
  
  \[
  \frac{\partial f}{\partial x}(x, y) = \frac{4}{3} x^{1/3} \sin(y/x) - y x^{-2/3} \cos(y/x),
  \]
  
  \[
  \frac{\partial f}{\partial y}(x, y) = x^{1/3} \cos(y/x),
  \]

  exist and are continuous in the open set
  
  \[ U = \{(x, y) \in \mathbb{R}^2 : x \neq 0\} . \]

  Therefore \( f \) is differentiable in \( U \).

- We claim that the partial derivatives of \( f \) also exist if \( x = 0 \) and are equal to 0.

- For the partial derivative with respect to \( x \), we have
  
  \[
  \frac{\partial f}{\partial x}(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \to 0} \frac{h^{4/3} \sin(y/h)}{h} = \lim_{h \to 0} h^{1/3} \sin(y/h) = 0 ,
  \]

  where we use the ‘squeeze’ theorem and the inequality
  
  \[ |h^{1/3} \sin(y/h)| \leq |h|^{1/3} . \]
• Since \( f(0, y) = 0 \) for every \( y \in \mathbb{R} \), for the partial derivative with respect to \( y \), we have
\[
\frac{\partial f}{\partial y}(0, y) = \frac{d}{dy} f(0, y) = 0.
\]

• It follows that if \( f \) is differentiable at \((0, y)\), then its derivative
\[
[df(0, y)] = (\partial f/\partial x(0, y), \partial f/\partial y(0, y))
\]
must be zero. We prove that this is indeed the case from the definition of the derivative.

• Consider the difference between \( f \) and its affine approximation at \((0, y)\):
\[
r(h, k) = f(h, y + k) - f(0, y) - df(0, y) \cdot (h, k).
\]
To prove that \( f \) is differentiable at \((0, y)\) with derivative \( df(0, y) \), we need to show that
\[
\lim_{(h,k) \to (0,0)} \frac{|r(h, k)|}{\| (h, k) \|} = 0.
\]

• Supposing that \( [df(0, y)] = (0, 0) \) and using the fact that \( f(0, y) = 0 \), we get \( r(h, k) = f(h, y + k) \). Therefore,
\[
|r(h, k)| \leq |h|^{4/3},
\]
since either \( h = 0 \) and \( r(h, k) = 0 \), or \( h \neq 0 \) and
\[
|r(h, k)| = |h^{4/3} \sin((y + k)/h)| \leq |h|^{4/3}.
\]
It follows that
\[
\frac{|r(h, k)|}{\| (h, k) \|} \leq \frac{|h|^{4/3}}{(h^2 + k^2)^{1/2}} \leq |h|^{1/3} \to 0 \quad \text{as } (h, k) \to (0, 0).
\]

• This proves that \( f \) is differentiable at \((0, y)\) with derivative \( df(0, y) = 0 \).

**Remark.** Note that \( \partial f/\partial y \) is continuous, but \( \partial f/\partial x \) is not continuous at \((0, y)\), except when \( y = 0 \). This function is differentiable even though it has a discontinuous partial derivative, so while the continuity of partial derivatives is a sufficient condition for differentiability, it isn’t a necessary one.
4. Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( x \in \mathbb{R}^n \). If \( A : \mathbb{R}^m \to \mathbb{R}^p \) is a linear map, prove from the definition of the derivative that \( Af : \mathbb{R}^n \to \mathbb{R}^p \) is differentiable at \( x \) and find its derivative. (You can assume that \( \|Ah\| \leq M\|h\| \) for some constant \( M \). See p. 212 of the text)

Solution.

- From the definition of the derivative,
  \[
  f(x + h) = f(x) + df(x)h + r(h)
  \]
  where the derivative \( df(x) : \mathbb{R}^n \to \mathbb{R}^m \) is a linear map and \( r(h) = o(h) \) as \( h \to 0 \), meaning that
  \[
  \lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0.
  \]

- It follows from the linearity of \( A \) that
  \[
  Af(x + h) = Af(x) + A df(x)h + Ar(h),
  \]
  and \( A df(x) : \mathbb{R}^n \to \mathbb{R}^p \) is a linear map (i.e., \( df(x) \) followed by \( A \)). Moreover, \( Ar(h) = o(h) \) as \( h \to 0 \) since
  \[
  \frac{\|Ar(h)\|}{\|h\|} \leq M \frac{\|r(h)\|}{\|h\|} \to 0 \quad \text{as} \quad h \to 0.
  \]

- This proves that \( Af \) is differentiable at \( x \) with derivative
  \[
  d(Af)(x) = A df(x).
  \]

Remark. This problem is a special case of the chain rule where one of the functions is linear, so the function and its derivative are equal.
5. The mean value theorem from one-variable calculus states that if a function \( f : [a, b] \to \mathbb{R} \) is continuous on the closed interval \([a, b]\) and differentiable in the open interval \((a, b)\), then there is a point \( a < c < b \) such that
\[
f(b) - f(a) = (b - a)f'(c).
\]
Does this theorem remain true for a vector-valued function \( f : [a, b] \to \mathbb{R}^2 \)?

Solution.

- It doesn’t remain true. The reason is that if \( f = (f_1, f_2) \), we may need to use different points \( c_1, c_2 \) to satisfy the mean value theorem for the real-valued component functions \( f_1, f_2 : [a, b] \to \mathbb{R} \).

- To give an explicit counter-example, define \( f : [0, 1] \to \mathbb{R}^2 \) by
\[
f(x) = (x(1 - x), x^2(1 - x)), \quad f_1(x) = x(1 - x), \quad f_2(x) = x^2(1 - x).
\]
Then \( f \) is continuous on \([0, 1]\) and differentiable in \((0, 1)\), since each component function is, and \( f(0) = f(1) = (0, 0) \). However, \( f_1'(c_1) = 0 \) if and only if \( c_1 = 1/2 \), while \( f_2'(c_2) = 0 \) at an interior point if and only if \( c_2 = 2/3 \). Thus, there is no point \( 0 < c < 1 \) such that \( f'(c) = (0, 0) \).

Optional remark. Frequently, we use the mean value theorem for a real-valued function in the following way: if \( |f'(x)| \leq M \) for \( a < x < b \), then \(|f(b) - f(a)| \leq M|b - a|\). Although the mean value theorem itself fails for vector-valued functions, there is a useful generalization of this estimate that can often be used instead. If \( f : \mathbb{R}^n \to \mathbb{R}^m \) is continuously differentiable, and \( a, b \in \mathbb{R}^n \), then the chain rule implies that for \( t \in \mathbb{R} \),
\[
\frac{d}{dt} f(a + t(b - a)) = df(a + t(b - a))(b - a).
\]
Suppose that \( \|df(a + t(b - a))\| \leq M \) for \( 0 \leq t \leq 1 \), where \( \|A\| \geq 0 \) denotes the norm of a linear map \( A : \mathbb{R}^n \to \mathbb{R}^m \) (i.e., the smallest constant such that \( \|Ah\| \leq \|A\|\|h\| \) for all \( h \in \mathbb{R}^n \)). Then the fundamental theorem of calculus implies that
\[
f(b) - f(a) = \int_0^1 \frac{d}{dt} f(a + t(b - a)) \, dt = \int_0^1 df(a + t(b - a))(b - a) \, dt,
\]
and this gives the estimate
\[
\|f(b) - f(a)\| \leq \int_0^1 \|df(a + t(b - a))(b - a)\| \, dt \leq M\|b - a\|.
\]