Problem 5.2.10 in the text.

2. Let \( K \subset [0,1] \) be the standard Cantor set described in class. Let \( f : [0,1] \to \mathbb{R} \) be the characteristic function of \( K \), defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \in K \\
0 & \text{if } x \notin K
\end{cases}
\]

(a) Prove that \( K \) is the set of discontinuities of \( f \).

(b) Prove that \( f \) is Riemann integrable. (Note: you’re not allowed to assume the Lebesgue criterion for Riemann integrability, but you can use the theorems about the existence of Riemann integrals that we proved in class.)

3. Suppose that \( F : [a,b] \to \mathbb{R} \) has a continuous derivative \( F' : [a,b] \to \mathbb{R} \). Use the second fundamental theorem of calculus (Theorem 5.3.3 in the text) to deduce the first fundamental theorem, that

\[
\int_a^b F' = F(b) - F(a).
\]

How is your conclusion weaker than the one stated in Theorem 5.3.1 of the text?

Sample Midterm Questions

Justify your answers with appropriate theorems.

1. Let

\[
L(x) = \int_1^x \frac{1}{t} dt.
\]

For what values of \( x \) is \( L(x) \) defined as an oriented Riemann integral?

2. Define \( f : [0,1] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
x & \text{if } x \in [0,1] \cap \mathbb{Q}, \\
0 & \text{if } x \in [0,1] \setminus \mathbb{Q}.
\end{cases}
\]
Compute the upper and lower integrals of $f$. Is $f$ Riemann integrable on $[0,1]$?

Hint. You can assume the summation formula
\[
\sum_{k=1}^{n} k = \frac{1}{2} n(n + 1).
\]

3. (a) State the second part of the fundamental theorem of calculus (i.e., the direction that says “the derivative of the integral is the original function”).

(b) Let $h : [-1, 1] \to \mathbb{R}$ be the Heaviside step function
\[
h(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases}
\]

Evaluate $H : [-1, 1] \to \mathbb{R}$ where
\[
H(x) = \int_{-1}^{x} h(x) \, dx.
\]

Is $H$ continuous on $[-1, 1]$? Is $H$ differentiable on $[-1, 1]$? Does $H$ satisfy the conclusions of the fundamental theorem of calculus you stated in (a)?

4. Define $F, f : [0, 1] \to \mathbb{R}$ by
\[
F(x) = x^2 \sin \left( \frac{1}{x} \right), \quad f(x) = -\cos \left( \frac{1}{x} \right) + 2x \sin \left( \frac{1}{x} \right)
\]
if $0 < x \leq 1$ and $F(0) = f(0) = 0$. Verify that that $F'(x) = f(x)$ for $0 < x < 1$ and evaluate
\[
\int_{0}^{1} f(x) \, dx.
\]

5. Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous positive function ($f \geq 0$) such that
\[
\int_{0}^{1} f = 0.
\]

Prove that $f = 0$. Does this result remain true if $f$ is only assumed to be integrable?