The \( \lim \inf \) and \( \lim \sup \) and Cauchy sequences

1 \hspace{1em} \textbf{The \( \lim \sup \) and \( \lim \inf \)}

We begin by stating explicitly some immediate properties of the sup and inf, which we use below.

**Proposition 1.** (a) If \( A \subset \mathbb{R} \) is a nonempty set, then \( \inf A \leq \sup A \). (b) If \( A \subset B \), then \( \sup A \leq \sup B \) and \( \inf A \geq \inf B \).

**Proof.** (a) If \( x \in A \), then \( \inf A \leq x \leq \sup A \), so the result follows. (b) If \( A \subset B \), then \( \sup B \) is an upper bound of \( A \), so \( \sup A \leq \sup B \). Similarly, \( \inf B \) is a lower bound of \( A \), so \( \inf A \geq \inf B \).

Suppose that \( (x_n) \) is a bounded sequence, meaning that there exist \( m, M \in \mathbb{R} \) such that

\[ m \leq x_n \leq M \quad \text{for all} \quad n \in \mathbb{N}. \]

Let \( T_n \subset \mathbb{R} \) be the set of terms of the tail of the sequence starting at \( x_n \),

\[ T_n = \{ x_k : k \geq n \}. \]

Then \( T_n \) is bounded from above by \( M \) and bounded from below by \( m \), so

\[ y_n = \sup T_n, \quad z_n = \inf T_n \]

exist, and

\[ m \leq z_n \leq y_n \leq M. \quad (1) \]

Moreover, \( T_{n+1} \subset T_n \), so \( y_{n+1} \leq y_n \) and \( z_{n+1} \geq z_n \). It follows that \( (y_n) \) is a decreasing sequence that is bounded from below by \( m \), and \( (z_n) \) is an increasing sequence that is bounded from above by \( M \), so both sequences converge. Their limits define the \( \lim \sup \) and \( \lim \inf \) of the original sequence.

**Definition 2.** Let \( (x_n) \) be a bounded sequence. Then

\[ \lim \sup x_n = \lim_{n \to \infty} \left[ \sup \{ x_k : k \geq n \} \right], \quad \lim \inf x_n = \lim_{n \to \infty} \left[ \inf \{ x_k : k \geq n \} \right]. \]

That is,

\[ \lim \sup x_n = \lim y_n, \quad \lim \inf x_n = \lim z_n. \]

It follows from (1) and the order properties of limits that

\[ \lim \inf x_n \leq \lim \sup x_n. \quad (2) \]
Theorem 3. A sequence \((x_n)\) converges to \(x \in \mathbb{R}\) if and only if
\[
\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = x.
\]

Proof. Suppose that the \(\limsup\) and \(\liminf\) of \((x_n)\) are both equal to \(x \in \mathbb{R}\). Then \(y_n \to x\) and \(z_n \to x\). The definition of \(y_n\) and \(z_n\) implies that \(z_n \leq x_n \leq y_n\) for every \(n \in \mathbb{N}\), so the “squeeze” theorem implies that \(x_n \to x\).

Conversely, suppose that \(x_n \to x\). Given any \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that
\[
x - \epsilon < x_n < x + \epsilon \quad \text{for every } n > N.
\]

It follows that
\[
x - \epsilon \leq \inf \{x_k : k \geq n\} \leq \sup \{x_k : k \geq n\} \leq x + \epsilon \quad \text{for every } n > N,
\]
which shows that
\[
|y_n - x| \leq \epsilon, \quad |z_n - x| \leq \epsilon \quad \text{for every } n > N.
\]

Hence, \(y_n \to x\) and \(z_n \to x\), so \(\limsup x_n = \liminf x_n = x\). \(\square\)

2 Cauchy sequences

A Cauchy sequence is a sequence whose terms eventually get arbitrarily close together.

Definition 4. A sequence \((x_n)\) of real numbers is a Cauchy sequence if for every \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that
\[
|x_m - x_n| < \epsilon \quad \text{for all } m, n > N.
\]

Every convergent sequence is Cauchy, and the completeness of \(\mathbb{R}\) implies that every Cauchy sequence converges.

Theorem 5. A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. First suppose that \((x_n)\) converges to a limit \(x \in \mathbb{R}\). Then for every \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that
\[
|x_n - x| < \frac{\epsilon}{2} \quad \text{for all } n > N.
\]

It follows that if \(m, n > N\), then
\[
|x_m - x_n| \leq |x_m - x| + |x - x_n| < \epsilon,
\]
which implies that \((x_n)\) is Cauchy. (This direction doesn’t use the completeness of \(\mathbb{R}\); for example, it holds equally well for sequence of rational numbers that converge in \(\mathbb{Q}\).)
Conversely, suppose that \((x_n)\) is Cauchy. Then there is \(N_1 \in \mathbb{N}\) such that
\[
|x_m - x_n| < 1 \quad \text{for all } m, n > N_1.
\]
It follows that if \(n > N_1\), then
\[
|x_n| \leq |x_n - x_{N_1+1}| + |x_{N_1+1}| \leq 1 + |x_{N_1+1}|.
\]
Hence the sequence is bounded with
\[
|x_n| \leq \max\{|x_1|, |x_2|, \ldots, |x_{N_1}|, 1 + |x_{N_1+1}|\}.
\]
Since the sequence is bounded, its lim sup and lim inf exist. We claim they are equal. Given \(\epsilon > 0\), choose \(N \in \mathbb{N}\) such that the Cauchy condition in Definition 4 holds. Then
\[
x_n - \epsilon < x_m < x_n + \epsilon \quad \text{for all } m \geq n > N.
\]
It follows that for all \(n > N\) we have
\[
x_n - \epsilon \leq \inf\{x_m : m \geq n\}, \quad \sup\{x_m : m \geq n\} \leq x_n + \epsilon,
\]
which implies that
\[
\sup\{x_m : m \geq n\} - \epsilon \leq \inf\{x_m : m \geq n\} + \epsilon.
\]
Taking the limit as \(n \to \infty\), we get that
\[
\limsup_{n \to \infty} x_n - \epsilon \leq \liminf_{n \to \infty} x_n + \epsilon,
\]
and since \(\epsilon > 0\) is arbitrary, we have
\[
\limsup_{n \to \infty} x_n \leq \liminf_{n \to \infty} x_n.
\]
In view of (2), it follows that \(\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n\), so the sequence converges by Theorem 3. \(\square\)