1.2. Some Preliminaries

which is precisely the desired conclusion \( x_{n+1} \leq x_{n+2} \). By induction, the claim is proved for all \( n \in \mathbb{N} \).

Any discussion about why induction is a valid argumentative technique immediately opens up a box of questions about how we understand the natural numbers. Earlier, in Section 1.1, we avoided this issue by referencing Kronecker’s famous comment that the natural numbers are somehow divinely given. Although we will not improve on this explanation here, it should be pointed out that a more atheistic and mathematically satisfying approach to \( \mathbb{N} \) is possible from the point of view of axiomatic set theory. This brings us back to a recurring theme of this chapter. Pedagogically speaking, the foundations of mathematics are best learned and appreciated in a kind of reverse order. A rigorous study of the natural numbers and the theory of sets is certainly recommended, but only after we have an understanding of the subtleties of the real number system. It is this latter topic that is the business of real analysis.

Exercises

Exercise 1.2.1. (a) Prove that \( \sqrt{3} \) is irrational. Does a similar argument work to show \( \sqrt{6} \) is irrational?

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove \( \sqrt{4} \) is irrational?

Exercise 1.2.2. Show that there is no rational number \( r \) satisfying \( 2^r = 3 \).

Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If \( A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots \) are all sets containing an infinite number of elements, then the intersection \( \bigcap_{n=1}^{\infty} A_n \) is infinite as well.

(b) If \( A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots \) are all finite, nonempty sets of real numbers, then the intersection \( \bigcap_{n=1}^{\infty} A_n \) is finite and nonempty.

(c) \( A \cap (B \cup C) = (A \cap B) \cup C \).

(d) \( A \cap (B \cap C) = (A \cap B) \cap C \).

(e) \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \).

Exercise 1.2.4. Produce an infinite collection of sets \( A_1, A_2, A_3, \ldots \) with the property that every \( A_i \) has an infinite number of elements, \( A_i \cap A_j = \emptyset \) for all \( i \neq j \), and \( \bigcup_{i=1}^{\infty} A_i = \mathbb{N} \).

Exercise 1.2.5 (De Morgan’s Laws). Let \( A \) and \( B \) be subsets of \( \mathbb{R} \).

(a) If \( x \in (A \cap B)^c \), explain why \( x \in A^c \cup B^c \). This shows that \( (A \cap B)^c \subseteq A^c \cup B^c \).
(b) Prove the reverse inclusion \((A \cap B)^c \supseteq A^c \cup B^c\), and conclude that \((A \cap B)^c = A^c \cup B^c\).

(c) Show \((A \cup B)^c = A^c \cap B^c\) by demonstrating inclusion both ways.

**Exercise 1.2.6.**

(a) Verify the triangle inequality in the special case where \(a\) and \(b\) have the same sign.

(b) Find an efficient proof for all the cases at once by first demonstrating \((a + b)^2 \leq (|a| + |b|)^2\).

(c) Prove \(|a - b| \leq |a - c| + |c - d| + |d - b|\) for all \(a, b, c,\) and \(d\).

(d) Prove \(||a| - |b|| \leq |a - b|\). (The unremarkable identity \(a = a - b + b\) may be useful.)

**Exercise 1.2.7.** Given a function \(f\) and a subset \(A\) of its domain, let \(f(A)\) represent the range of \(f\) over the set \(A\); that is, \(f(A) = \{f(x) : x \in A\}\).

(a) Let \(f(x) = x^2\). If \(A = [0, 2]\) (the closed interval \(\{x \in \mathbb{R} : 0 \leq x \leq 2\}\)) and \(B = [1, 4]\), find \(f(A)\) and \(f(B)\). Does \(f(A \cap B) = f(A) \cap f(B)\) in this case? Does \(f(A \cup B) = f(A) \cup f(B)\)?

(b) Find two sets \(A\) and \(B\) for which \(f(A \cap B) \neq f(A) \cap f(B)\).

(c) Show that, for an arbitrary function \(g : \mathbb{R} \to \mathbb{R}\), it is always true that \(g(A \cap B) \subseteq g(A) \cap g(B)\) for all sets \(A, B \subseteq \mathbb{R}\).

(d) Form and prove a conjecture about the relationship between \(g(A \cup B)\) and \(g(A) \cup g(B)\) for an arbitrary function \(g\).

**Exercise 1.2.8.** Here are two important definitions related to a function \(f : A \to B\). The function \(f\) is one-to-one (1-1) if \(a_1 \neq a_2\) in \(A\) implies that \(f(a_1) \neq f(a_2)\) in \(B\). The function \(f\) is onto if, given any \(b \in B\), it is possible to find an element \(a \in A\) for which \(f(a) = b\).

Give an example of each or state that the request is impossible:

(a) \(f : \mathbb{N} \to \mathbb{N}\) that is 1-1 but not onto.

(b) \(f : \mathbb{N} \to \mathbb{N}\) that is onto but not 1-1.

(c) \(f : \mathbb{N} \to \mathbb{Z}\) that is 1-1 and onto.

**Exercise 1.2.9.** Given a function \(f : D \to \mathbb{R}\) and a subset \(B \subseteq \mathbb{R}\), let \(f^{-1}(B)\) be the set of all points from the domain \(D\) that get mapped into \(B\); that is, \(f^{-1}(B) = \{x \in D : f(x) \in B\}\). This set is called the preimage of \(B\).

(a) Let \(f(x) = x^2\). If \(A\) is the closed interval \([0, 4]\) and \(B\) is the closed interval \([-1, 1]\), find \(f^{-1}(A)\) and \(f^{-1}(B)\). Does \(f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)\) in this case? Does \(f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)\)?
1.2. Some Preliminaries

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function \( g : \mathbb{R} \to \mathbb{R} \), it is always true that \( g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B) \) and \( g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B) \) for all sets \( A, B \subseteq \mathbb{R} \).

Exercise 1.2.10. Decide which of the following are true statements. Provide a short justification for those that are valid and a counterexample for those that are not:

(a) Two real numbers satisfy \( a < b \) if and only if \( a < b + \epsilon \) for every \( \epsilon > 0 \).
(b) Two real numbers satisfy \( a < b \) if \( a < b + \epsilon \) for every \( \epsilon > 0 \).
(c) Two real numbers satisfy \( a < b \) if and only if \( a < b + \epsilon \) for every \( \epsilon > 0 \).

Exercise 1.2.11. Form the logical negation of each claim. One trivial way to do this is to simply add “It is not the case that...” in front of each assertion. To make this interesting, fashion the negation into a positive statement that avoids using the word “not” altogether. In each case, make an intuitive guess as to whether the claim or its negation is the true statement.

(a) For all real numbers satisfying \( a < b \), there exists an \( n \in \mathbb{N} \) such that \( a + 1/n < b \).
(b) There exists a real number \( x > 0 \) such that \( x < 1/n \) for all \( n \in \mathbb{N} \).
(c) Between every two distinct real numbers there is a rational number.

Exercise 1.2.12. Let \( y_1 = 6 \), and for each \( n \in \mathbb{N} \) define \( y_{n+1} = (2y_n - 6)/3 \).

(a) Use induction to prove that the sequence satisfies \( y_n > -6 \) for all \( n \in \mathbb{N} \).
(b) Use another induction argument to show the sequence \( (y_1, y_2, y_3, \ldots) \) is decreasing.

Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

\[
(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c
\]

for any finite \( n \in \mathbb{N} \).

(b) It is tempting to appeal to induction to conclude

\[
(\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c,
\]

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of \( n \in \mathbb{N} \), but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets \( B_1, B_2, B_3, \ldots \) where \( \bigcap_{i=1}^{\infty} B_i \neq \emptyset \) is true for every \( n \in \mathbb{N} \), but \( \bigcap_{i=1}^{\infty} B_i \neq \emptyset \) fails.