MAT 127B-01 Winter 05 Midterm 1 Answers

1.(20 pts) Consider the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n} x^{2n}$$

a. Compute its radius R of convergence.
b. Does it converge at $x = \pm R$, explain your answer.

Answer:
a. Let $y = x^2$. The new power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n} y^n$$

has radius of convergence that is $\frac{1}{\beta}$ where

$$\beta = \limsup \left| \frac{3^n}{n} \right|^{\frac{1}{n}} = 3$$

Hence the series converges for $|y| < \frac{1}{3}$ or equivalently $|x^2| < \frac{1}{3}$ or $|x| < \sqrt{\frac{1}{3}}$. So $R = \sqrt{\frac{1}{3}}$.
b. At $x = \pm R = \pm\sqrt{\frac{1}{3}}$ the series is

$$\sum_{n=1}^{\infty} \frac{3^n}{n} \left(\pm\sqrt{\frac{1}{3}} \right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges.

2.(20 pts) Consider the sequence of functions $f_n(x) = x - x^n$.
a. Does this sequence converge pointwise on $[0, 1]$? If so what is the limit function?
b. Does this sequence converge uniformly on $[0, 1]$? Prove your answer.
c. Does this sequence converge uniformly on $[0, a]$ where $0 < a < 1$? Prove your answer.

Answer:
a.

$$\lim_{n\to\infty} = f(x) = \begin{cases} x & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

b. The convergence cannot be uniform on $[0, 1]$ because the pointwise limit $f(x)$ is not continuous on $[0, 1]$.
c.

$$\sup_{[0,a]} |f(x) - f_n(x)| = \sup_{[0,a]} |x^n| = a^n$$

which goes to zero as $n \to \infty$. Hence the convergence is uniform on $[0, a]$.

3. (20 pts) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ converges to a continuous function on $(-\infty, \infty)$.

Answer:
We use the Weierstrass M test.

$$|\frac{\sin nx}{n^3}| \leq \frac{1}{n^3} = M_n$$

and

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$

Hence the series converges uniformly on $(-\infty, \infty)$ and so the limiting function is continuous.

4.(20 pts) Suppose $f_n(x)$ converges uniformly on $[a, b]$ to a bounded function $f(x)$ and $g_n(x)$ converges uniformly on $[a, b]$ to a bounded function $g(x)$. Show that $h_n(x) = f_n(x)g_n(x)$ converges uniformly to $h(x) = f(x)g(x)$.

Answer:
Suppose $f(x)$ and $g(x)$ are bounded by M. Given any $0 < \epsilon < M$ there exists an N such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ and $|g_n(x) - g(x)| < \epsilon$. If $n > N$ then $|f_n(x)| \leq |f(x)| + |f_n(x) - f(x)| < M + \epsilon < 2M$. Now

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)|$$

so

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)|$$

$$|f_n(x)g_n(x) - f(x)g(x)| < 2M\epsilon + M\epsilon = 3M\epsilon$$

5.(20 pts) Find the power series of the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

that satisfies the differential equation

$$f'(x) = xf(x) + 1, \qquad f(0) = 0$$

Answer:
Since $f(0) = 0$ we have $a_0 = 0$. We substitute the power series into the differential equation and obtain

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1} + 1$$

or equivalently

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{k=1}^{\infty} a_{k-1} x^k + 1$$

If we equate the coefficients of like powers of x we obtain that $a_1 = 1$ and for $k > 0$

$$a_{k+1} = \frac{1}{k+1} a_{k-1}$$

Hence we conclude that

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{1}{k(k-2)(k-4)\cdots 1} & k \text{ odd} \end{cases}$$