1. (20 pts) Consider the power series
\[ \sum_{n=1}^{\infty} \frac{3^n}{n} x^{2n} \]

a. Compute its radius \( R \) of convergence.
b. Does it converge at \( x = \pm R \), explain your answer.

Answer:
a. Let \( y = x^2 \). The new power series
\[ \sum_{n=1}^{\infty} \frac{3^n}{n} y^n \]
has radius of convergence that is \( \frac{1}{\beta} \) where
\[ \beta = \limsup \left| \frac{3^n}{n} \right|^{\frac{1}{n}} = 3 \]
Hence the series converges for \( |y| < \frac{1}{3} \) or equivalently \( |x^2| < \frac{1}{3} \) or \( |x| < \sqrt{\frac{1}{3}} \).
So \( R = \sqrt{\frac{1}{3}} \).
b. At \( x = \pm R = \pm \sqrt{\frac{1}{3}} \) the series is
\[ \sum_{n=1}^{\infty} \frac{3^n}{n} \left( \pm \sqrt{\frac{1}{3}} \right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n} \]
which diverges.
2. (20 pts) Consider the sequence of functions $f_n(x) = x - x^n$.

a. Does this sequence converge pointwise on $[0, 1]$? If so what is the limit function?

b. Does this sequence converge uniformly on $[0, 1]$? Prove your answer.

c. Does this sequence converge uniformly on $[0, a]$ where $0 < a < 1$? Prove your answer.

Answer:

a. 

$$\lim_{n \to \infty} f(x) = \begin{cases} x & x \in [0, 1) \\ 0 & x = 1 \end{cases}$$

b. The convergence cannot be uniform on $[0, 1]$ because the pointwise limit $f(x)$ is not continuous on $[0, 1]$.

c. 

$$\sup_{[0,a]} |f(x) - f_n(x)| = \sup_{[0,a]} |x^n| = a^n$$

which goes to zero as $n \to \infty$. Hence the convergence is uniform on $[0, a]$. 


3. (20 pts) Show that \( \sum_{n=1}^{\infty} \frac{\sin nx}{n^3} \) converges to a continuous function on \((-\infty, \infty)\).

Answer:
We use the Weierstrass M test.

\[
\left| \frac{\sin nx}{n^3} \right| \leq \frac{1}{n^3} = M_n
\]

and

\[
\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty
\]

Hence the series converges uniformly on \((-\infty, \infty)\) and so the limiting function is continuous.
4. (20 pts) Suppose $f_n(x)$ converges uniformly on $[a, b]$ to a bounded function $f(x)$ and $g_n(x)$ converges uniformly on $[a, b]$ to a bounded function $g(x)$. Show that $h_n(x) = f_n(x)g_n(x)$ converges uniformly to $h(x) = f(x)g(x)$.

Answer:

Suppose $f(x)$ and $g(x)$ are bounded by $M$. Given any $0 < \epsilon < M$ there exists an $N$ such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ and $|g_n(x) - g(x)| < \epsilon$. If $n > N$ then $|f_n(x)| \leq |f(x)| + |f_n(x) - f(x)| < M + \epsilon < 2M$. Now

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)|$$

so

$$|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)|$$

$$|f_n(x)g_n(x) - f(x)g(x)| < 2M\epsilon + M\epsilon = 3M\epsilon$$
5. (20 pts) Find the power series of the function

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]

that satisfies the differential equation

\[ f'(x) = xf(x) + 1, \quad f(0) = 0 \]

Answer:

Since \( f(0) = 0 \) we have \( a_0 = 0 \). We substitute the power series into the differential equation and obtain

\[ \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1} + 1 \]

or equivalently

\[ \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{k=1}^{\infty} a_k x^k + 1 \]

If we equate the coefficients of like powers of \( x \) we obtain that \( a_1 = 1 \) and for \( k > 0 \)

\[ a_{k+1} = \frac{1}{k+1} a_k \]

Hence we conclude that

\[ a_k = \begin{cases} 0 & \text{k even} \\ \frac{1}{k(k-2)(k-4)\cdots} & \text{k odd} \end{cases} \]