MAT 127B-01 Winter 05 Midterm 2

1.(20 pts) Let

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a. Is f continuous at $x = 0$? Explain.
b. Is f differentiable at $x \neq 0$. Explain your answer and compute $f'(x)$ if possible.
c. Is f differentiable at $x = 0$. Explain your answer and compute $f'(x)$ if possible.

Answer: a. f is continuous at $x = 0$ because

$$|f(x) - f(0)| = |x \sin \frac{1}{x}| \leq |x| \to 0$$

as $x \to 0$.
b. At $x \neq 0$, f is composition and product of differentiable functions. Its derivative can be computed using the chain rule and product formula.

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

c. We use the definition of the derivative

$$f'(0) = \lim_{x \to 0} \frac{x \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$$

which does not exist.

2.(20 pts) Suppose that f is differentiable and $2 \leq f'(x) \leq 3$ on $\mathbb{R}$. Show that for $x \geq 0$

$$2x \leq f(x) - f(0) \leq 3x$$

Answer:
Let $g(x) = f(x) - 2x - f(0)$ and $h(x) = 3x + f(0) - f(x)$ then

$$g'(x) = f'(x) - 2 \geq 0$$
$$h(x) = 3 - f'(x)$$

so $g,\ h$ are increasing functions. If $x \geq 0$ then $g(x) = f(x) - 2x - f(0) \geq g(0) = 0$ and $h(x) = 3x + f(0) - f(x) \geq h(0) = 0$.

3.(20 pts) Find the following limits if they exist.

$$\lim_{x\to 0} \quad \tfrac{\sin x - x\cos x}{x - \sin x}$$

$$\lim_{x\to 1} \quad x^{\frac{1}{1-x}}$$

Answer: By repeated application of L'Hospital's rule

$$\begin{aligned} &\lim_{x\to 0} \quad \tfrac{\sin x - x\cos x}{x - \sin x} \\ &= \lim_{x\to 0} \quad \tfrac{x\sin x}{1-\cos x} \\ &= \lim_{x\to 0} \quad \tfrac{\sin x + x\cos x}{\sin x} \\ &= \lim_{x\to 0} \quad \tfrac{2\cos x - x\sin x}{\cos x} \\ &= 2 \end{aligned}$$

$$\begin{aligned} &\lim_{x\to 1} \quad x^{\frac{1}{1-x}} \\ &= \lim_{x\to 1} \quad \exp \tfrac{\log x}{1-x} \\ &= \lim_{x\to 1} \quad \exp \tfrac{-1}{x} \\ &= e^{-1} \end{aligned}$$

4.(20 pts)
a. Suppose $f(x) = \sqrt{1+x}$ show that

$$f^{(k)}(x) = \frac{(-1)^{k-1}}{2^k} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)(1+x)^{\frac{1-2k}{2}}$$

b. Find the Taylor series of $f(x)$ where the remainder is of degree n.
c. What is the remainder $R_n(x)$.

Answer:

a.

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{\frac{-1}{2}} \\ f''(x) &= -\frac{1}{4}(1+x)^{\frac{-3}{2}} \end{aligned}$$

Suppose by induction that

$$f^{(k)}(x) = \frac{(-1)^{k-1}}{2^k} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)(1+x)^{\frac{1-2k}{2}}$$

then

$$\begin{aligned} f^{(k+1)}(x) &= -\frac{2k-1}{2}\frac{(-1)^{k-1}}{2^k} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)(1+x)^{\frac{1-2(k+1)}{2}} \\ &= \frac{(-1)^{k}}{2^{k+1}} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)(2k-1)(1+x)^{\frac{1-2(k+1)}{2}} \end{aligned}$$

b.

$$f^{(k)}(0) = \frac{(-1)^{k-1}}{2^k} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)$$

so

$$f(x) = \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{2^k} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)}{k!} x^k + R_n(x)$$

c.

$$R_n(x) = \frac{(-1)^{n-1}}{2^n} 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)(1+y)^{\frac{1-2n}{2}}$$

where y is between 0 and x.

5.(20 pts) Assume f and f' are differentiable and f'' is continuous on $\mathbb{R}$. Use L'Hospital's rule to show

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Justify your steps.

Answer:
Since f is differentiable on $I\!R$ so it is continuous,

$$\lim_{h \to 0} f(x+h) - 2f(x) + f(x-h) = 0$$

and of course

$$\lim_{h \to 0} h^2 = 0$$

so we can apply L'Hospital's rule. We differentiate with respect to h as this is the variable that is going to zero and we obtain

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

Since f' is differentiable on $I\!R$ so it is continuous,

$$\lim_{h \to 0} f'(x+h) - f'(x-h) = 0$$

and of course

$$\lim_{h \to 0} 2h = 0$$

so we can apply L'Hospital's rule again. Differentiating with respect to h we obtain

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2} = f''(x)$$

because f'' is continuous.