1. Define \( f_n, g_n : [0, 1] \to \mathbb{R} \) by

\[
\begin{align*}
  f_n(x) &= \frac{nx^2}{1 + n^2x^2}, \\
  g_n(x) &= \frac{n^2x}{1 + n^2x^2}.
\end{align*}
\]

Show that the sequences \((f_n), (g_n)\) converge pointwise on \([0, 1]\), and determine their pointwise limits. Determine (with proof) whether or not each sequence converges uniformly on \([0, 1]\).

Solution.

• As \( n \to \infty \), we have \( f_n \to 0 \) and \( g_n \to g \) pointwise, where

\[
g(x) = \begin{cases} 
  \frac{1}{x} & \text{if } 0 < x \leq 1, \\
  0 & \text{if } x = 0.
\end{cases}
\]

• Given \( \epsilon > 0 \), choose \( N = 1/\epsilon \). Then \( n > N \) implies that

\[
|f_n(x)| = \frac{1}{n} \left( \frac{nx^2}{1/n + nx^2} \right) \leq \frac{1}{n} < \epsilon \quad \text{for all } x \in [0, 1].
\]

Therefore \( f_n \) converges uniformly to 0.

• The functions \( g_n \) are continuous, and their pointwise limit \( g \) is discontinuous. Since the uniform limit of continuous functions is continuous, \((g_n)\) does not converge uniformly.
2. Find all points \( x \in \mathbb{R} \) where the following power series converges:

\[
\sum_{n=0}^{\infty} \frac{1}{1 + n2^n} x^n.
\]

Solution.

- According to the ratio test, the radius of convergence \( R \) of the power series \( \sum a_n x^n \) is given by

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|
\]

(provided that this limit exists). Hence the radius of convergence of the given power series is

\[
R = \lim_{n \to \infty} \frac{1 + (n + 1)2^{n+1}}{1 + n2^n} = \lim_{n \to \infty} \frac{1/(n2^n) + (1 + 1/n)2}{1/(n2^n) + 1} = 2.
\]

- When \( x = 2 \), the series is

\[
\sum_{n=0}^{\infty} \frac{2^n}{1 + n2^n} = \sum_{n=0}^{\infty} \frac{1}{n + 2^{-n}}.
\]

Since

\[
\frac{1}{n + 2^{-n}} \geq \frac{1}{n + 1}
\]

this series diverges by comparison with the divergent harmonic series

\[
\sum_{n=0}^{\infty} \frac{1}{n + 1}.
\]

- When \( x = -2 \), the series is

\[
\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{1 + n2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 2^{-n}}.
\]

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which converges by the alternating series test, since
\[
\frac{1}{n + 2^{-n}} \to 0 \quad \text{as } n \to \infty
\]
and is decreasing in $n$.

- The power series therefore converges for $-2 \leq x < 2$. 
3. (a) Prove that the following series converge on $\mathbb{R}$ to continuous functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$ 

(b) Prove that $g$ is differentiable on $\mathbb{R}$, and $g' = f$.

Solution.

- (a) Since
  $$\left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2}, \quad \left| \frac{\sin nx}{n^3} \right| \leq \frac{1}{n^3}$$
  for all $x \in \mathbb{R}$ and
  $$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$
  the Weierstrass $M$-test implies that both series converge uniformly on $\mathbb{R}$. Since the terms in the series are continuous, and the uniform limit of continuous functions is continuous, the sums $f, g$ are continuous.

- (b) Since the uniform convergence of Riemann integrable functions implies convergence of their Riemann integrals, we can integrate the series for $f$ term-by-term over the interval $[0, x]$ (or $[x, 0]$ if $x < 0$) to obtain

$$\int_0^x f(t) \, dt = \sum_{n=1}^{\infty} \int_0^x \frac{\cos nt}{n^2} \, dt = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3} = g(x).$$

Since $f$ is continuous, the fundamental theorem of calculus implies that $g$ is differentiable and $g' = f$. 

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4. Let \(a > 0\). Give a definition of the following improper Riemann integral as a limit of Riemann integrals:

\[
\int_2^\infty \frac{1}{x(\log x)^a} \, dx.
\]

For what values of \(a\) does this integral converge?

**Solution.**

- We define
  \[
  \int_2^\infty \frac{1}{x(\log x)^a} \, dx = \lim_{b \to \infty} \int_2^b \frac{1}{x(\log x)^a} \, dx.
  \]

- Let
  \[I(b) = \int_2^b \frac{1}{x(\log x)^a} \, dx.\]

Making the substitution \(u = \log x\), we get

\[
I(b) = \int_{\log 2}^{\log b} \frac{1}{u^a} \, du.
\]

For \(a \neq 1\), we have

\[
I(b) = \left[ \frac{u^{1-a} \log b}{1 - a} \right]_{\log 2}^{\log b}
\]

\[= \frac{(\log b)^{1-a} - (\log 2)^{1-a}}{1 - a},\]

which diverges as \(b \to \infty\) if \(a < 1\). If \(a > 1\), then

\[I(b) \to \frac{(\log 2)^{1-a}}{a - 1} \quad \text{as } b \to \infty.\]

If \(a = 1\), then

\[
I(b) = \left[ \log u \right]_{\log 2}^{\log b}
\]

\[= \log(\log b) - \log(\log 2)
\]

\[\to \infty \quad \text{as } b \to \infty.\]

- The improper integral therefore converges when \(a > 1\), and then

\[
\int_2^\infty \frac{1}{x(\log x)^a} \, dx = \frac{(\log 2)^{1-a}}{a - 1}.
\]
5. Define \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
  x & \text{if } x \in \mathbb{Q}, \\
  0 & \text{if } x \notin \mathbb{Q}.
\end{cases}
\]

Is \( f \) Riemann integrable on \([0, 1]\)? Prove your answer.

**Solution.**

- The function \( f \) is not Riemann integrable.

- Suppose that \( P = \{t_0, t_1, \ldots, t_n\} \) is any partition of \([0, 1]\) (so \( t_0 = 0, t_n = 1, \) and \( t_{k-1} < t_k \)). Since every interval \([t_{k-1}, t_k]\) contains irrational numbers, we have

\[
m(f, [t_{k-1}, t_k]) = \inf \{ f(x) : x \in [t_{k-1}, t_k] \} = 0.
\]

The lower Darboux sum of \( f \) is therefore given by

\[
L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) (t_k - t_{k-1}) = 0,
\]

and the lower Darboux integral of \( f \) is

\[
L(f) = \sup \{ L(f, P) : P \text{ is a partition of } [0, 1] \} = 0.
\]

- Since the rational numbers are dense in any interval, we have

\[
M(f, [t_{k-1}, t_k]) = \sup \{ f(x) : x \in [t_{k-1}, t_k] \} = t_k.
\]

Define \( \ell : [0, 1] \to \mathbb{R} \) by \( \ell(x) = x \). Then

\[
U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) (t_k - t_{k-1})
= \sum_{k=1}^{n} t_k (t_k - t_{k-1})
= U(\ell, P).
\]

Therefore

\[
U(f) = \inf \{ U(f, P) : P \text{ is a partition of } [0, 1] \} = U(\ell).
\]
Since $\ell$ is Riemann integrable,

$$U(\ell) = \int_0^1 x \, dx = \frac{1}{2}.$$  

So $U(f) = 1/2$. Thus $U(f) > L(f)$, and $f$ is not Riemann integrable.
6. Suppose that

\[ F(x) = \begin{cases} 
  -x^2 & \text{for } -1 \leq x < 0, \\
  x^2 + 2 & \text{for } 0 \leq x \leq 1. 
\end{cases} \]

Evaluate the Riemann-Stieltjes integral

\[ \int_{-1}^{1} e^{x^2} dF(x). \]

Briefly justify your computations.

Solution.

\begin{itemize}
  \item We write \( F = F_1 + F_2 \), where
    \[
    F_1(x) = \begin{cases} 
      0 & \text{for } -1 \leq x < 0, \\
      2 & \text{for } 0 \leq x \leq 1,
    \end{cases}
    \]
    \[
    F_2(x) = \begin{cases} 
      -x^2 & \text{for } -1 \leq x < 0, \\
      x^2 & \text{for } 0 \leq x \leq 1.
    \end{cases}
    \]
  \item Using standard properties of the Riemann-Stieltjes integral, and its expression for jump and continuously differentiable integrators, we get
    \[
    \int_{-1}^{1} e^{x^2} dF(x) = \int_{-1}^{1} e^{x^2} dF_1(x) + \int_{-1}^{1} e^{x^2} dF_2(x)
    \]
    \[
    = \int_{-1}^{1} e^{x^2} dF_1(x) + \int_{-1}^{0} e^{x^2} dF_2(x) + \int_{0}^{1} e^{x^2} dF_2(x)
    \]
    \[
    = e^0 \cdot 2 + \int_{-1}^{0} e^{x^2} d(-x^2) + \int_{0}^{1} e^{x^2} d(x^2)
    \]
    \[
    = 2 - \int_{-1}^{0} 2xe^{x^2} dx + \int_{0}^{1} 2xe^{x^2} dx
    \]
    \[
    = 2 - \left[ e^{x^2} \right]_{-1}^{0} + \left[ e^{x^2} \right]_{0}^{1}
    \]
    \[
    = 2 - (1 - e) + (e - 1)
    \]
    \[
    = 2e.
    \]
\end{itemize}
7. (a) Find the Taylor series of $e^{-x}$ (at $x = 0$).
(b) Give an expression for the remainder $R_n(x)$ between $e^{-x}$ and its Taylor polynomial of degree $n - 1$ involving an intermediate point $y$ between 0 and $x$.
(c) Prove from your expression in (b) that the Taylor series for $e^{-x}$ converges to $e^{-x}$ for every $x \in \mathbb{R}$. (Don’t use general theorems.)

Solution.

• (a) Let $f(x) = e^{-x}$. Then

$$
f^{(k)}(x) = (-1)^k e^{-x}.
$$

The $k$th Taylor coefficient of $f$ is

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(-1)^k}{k!}.
$$

The Taylor series of $e^{-x}$ is therefore

$$
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = 1 - x + \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \ldots
$$

• (b) By the Taylor remainder theorem,

$$
e^{-x} = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} x^k + R_n(x),
$$

where

$$R_n(x) = \frac{(-1)^n e^{-y}}{n!} x^n
$$

for some $y$ between 0 and $x$.

• (c) If $x > 0$, then $0 < y < x$ and $e^{-y} < 1$. Hence

$$|R_n(x)| < \frac{x^n}{n!} \to 0 \quad \text{as } n \to \infty.
$$

(Note that if $c_n = x^n/n!$ then $c_{n+1}/c_n = x/(n+1) < 1/2$ for $n > 2x$, so $c_n \to 0$ as $n \to \infty$ for every $x > 0$.) Taking the limit as $n \to \infty$ in (1), we obtain that

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k.$$
If $x < 0$, then $e^{-y} < e^{-x}$, and the Taylor series also converges, since

$$|R_n(x)| < e^{-x} \frac{|x|^n}{n!} \to 0 \quad \text{as } n \to \infty.$$
8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \left[ \sin\left(\frac{1}{x}\right) - 2 \right] & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Prove that $f(x)$ has a strict maximum at $x = 0$ (i.e. $f(0) > f(x)$ for all $x \neq 0$).

(b) Prove that $f$ is differentiable on $\mathbb{R}$.

(c) Prove that $f$ is not increasing on the interval $(-\epsilon, 0)$ and $f$ is not decreasing on the interval $(0, \epsilon)$ for any $\epsilon > 0$.

Solution.

- (a) We have $f(0) = 0$. If $x \neq 0$, then since $\sin(1/x) \leq 1$

$$f(x) \leq x^2 \left[1 - 2 \right] \leq -x^2 < 0.$$

- (b) The function $f$ is differentiable at any nonzero $x$ since it is a product and composition of differentiable functions. At $x = 0$ the function is differentiable, with $f'(0) = 0$, since

$$\lim_{x \to 0} \left\{ \frac{f(x) - f(0)}{x - 0} \right\} = \lim_{x \to 0} \left\{ x \left[ \sin \left( \frac{1}{x} \right) - 2 \right] \right\} = 0.$$

- (c) For $x \neq 0$, we compute using the chain and product rules that

$$f'(x) = -\cos \left( \frac{1}{x} \right) + 2x \left[ \sin \left( \frac{1}{x} \right) - 2 \right].$$

If $|x| \leq 1/12$ then

$$\left[ 2x \left[ \sin \left( \frac{1}{x} \right) - 2 \right] \right] \leq 6|x| < \frac{1}{2},$$

so

$$- \cos \left( \frac{1}{x} \right) - \frac{1}{2} < f'(x) < - \cos \left( \frac{1}{x} \right) + \frac{1}{2}.$$ It follows that $f' < 0$ (hence $f$ is strictly decreasing) in any interval where $\cos(1/x) > 1/2$, and $f' > 0$ (hence $f$ is strictly increasing) in any interval where $\cos(1/x) < -1/2$. Since there exist such intervals arbitrarily close to 0, the function $f$ is not increasing throughout any interval $(-\epsilon, 0)$, nor is it decreasing throughout any interval $(0, \epsilon)$. 

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• This example shows that a differentiable function may attain a maximum at a point even though it’s not increasing on any interval to the left of the point or decreasing on any interval to the right.