Sample Questions: Solutions
Midterm II
Math 127B. Winter, 2005

1. For each of the following statements, say if it is true or false. (No explanation is required.)

(a) If f is differentiable and $f' > 0$, then f is strictly increasing.

(b) If f is strictly increasing and differentiable, then $f' > 0$.

(c) If f is the sum of a convergent Taylor series in an open interval containing the origin, then f is infinitely differentiable.

(d) If f is infinitely differentiable in an open interval containing the origin, then the Taylor series of f converges.

(e) There exists $0 < x < 1$ such that $e^x \sin 1 = \cos x \, (e - 1)$.

Solution.

- (a) True. (Follows from the mean value theorem.)
- (b) False. (For example, $f(x) = x^3$.)
- (c) True. (Power series can be differentiated term-by-term inside their interval of convergence.)
- (d) False. (For example, we saw in class that

$$f(x) = \sum_{n=1}^{\infty} e^{-n} \cos(n^2 x)$$

is infinitely differentiable on $\mathbb{R}$, but its Taylor series (at 0) diverges.)
- (e) True. (Apply the generalized mean value theorem to e^x and $\sin x$ on $[0, 1]$.)

2. Define the derivative. Consider

$$f(x) = \begin{cases} |x|^a & \text{for } x \text{ irrational,} \\ 0 & \text{for } x \text{ rational.} \end{cases}$$

For what values of $a > 0$ is f differentiable at 0? Is f differentiable at $x \neq 0$?

Solution.

- A function $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ with derivative $f'(x_0)$ if the following limit exists

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

- This function is differentiable at 0 if and only if $a > 1$. This follows because

$$\frac{f(x) - f(0)}{x - 0} = 0 \qquad \text{if } x \text{ is rational and nonzero,}$$
$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|^a}{x} \qquad \text{if } x \text{ is irrational,}$$

  and

$$\lim_{x \to 0} \frac{|x|^a}{x} = 0$$

  if and only if $a > 1$.

- The function f is discontinuous at every $x \neq 0$, therefore it is not differentiable at any $x \neq 0$.

3. State Taylor's theorem. Prove that

$$\log(1+x) < x$$

for all $x > 0$.

Solution.

- If $f : (a,b) \to \mathbb{R}$ is an n-times differentiable function on an open interval (a,b) containing the origin, then for any $x \in (a,b)$ there exists a y between 0 and x such that

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k = \frac{f^{(n)}(y)}{n!} x^n.$$

- First proof. The function $f(x) = \log(1+x)$ is infinitely differentiable in $(-1, \infty)$, and

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}.$$

Since $f(0) = 0$ and $f'(0) = 1$, the Taylor polynomial of f of degree 1 is x. If $x > 0$, then using Taylor's theorem with $n = 2$ we find that there exists $0 < y < x$ such that

$$f(x) - x = -\frac{1}{2!(1+y)^2} x^2 < 0,$$

so

$$\log(1+x) < x.$$

- Second proof. For any $x > 0$, the function $g(x) = x - \log(1+x)$ is continuous on the closed interval $[0,x]$ and differentiable in the open interval $(0,x)$. By the mean value theorem, there exists $0 < y < x$ such that

$$g(x) - g(0) = g'(y)(x - 0).$$

Since $g(0) = 0$ and

$$g'(y) = 1 - \frac{1}{1+y} > 0 \qquad \text{for } y > 0,$$

we conclude that

$$g(x) > 0 \qquad \text{for } x > 0$$

which proves the result.

4. Carefully state a version of L'Hospital's rule that applies to the following limit. Use it to prove that the limit exists, and find its value:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

Solution.

- Suppose that f, g are functions that are differentiable in an open interval containing x_0, with $g'(x)$ nonzero in the interval, and

$$\lim_{x\to x_0} f(x) = 0, \qquad \lim_{x\to x_0} g(x) = 0.$$

If the limit

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$$

exists, then so does the limit

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = L.$$

- The functions $f(x) = \sin x$ and $g(x) = 2x$ satisfy the hypotheses of L'Hospital's theorem on any interval containing the origin, and

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

since $\cos x$ is continuous at $x = 0$. Hence

$$\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

Applying L'Hospital's theorem again to the functions $F(x) = 1 - \cos x$ and $G(x) = x^2$, which also satisfy its hypotheses, with

$$\lim_{x\to 0} \frac{F'(x)}{G'(x)} = \frac{1}{2}$$

we conclude that

$$\lim_{x\to 0} \frac{F(x)}{G(x)} = \frac{1}{2},$$

which proves the result.

5. Define the hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Prove that $\sinh x$ is strictly increasing on $\mathbb{R}$ and hence has an inverse. Prove that the inverse is differentiable and compute its derivative.

Solution.

- The functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

  satisfy the following easily verified identities:

$$\begin{aligned} &(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x, \\ &\cosh^2 x - \sinh^2 x = 1. \end{aligned}$$

- Since $(\sinh x)' = \cosh x > 0$ on $\mathbb{R}$, the function $\sinh x$ is strictly increasing, and therefore invertible on its range. Moreover, since $e^x \to \infty$ and $e^{-x} \to 0$ as $x \to \infty$, we have $\sinh x \to \infty$ as $x \to \infty$; similarly $\sinh x \to -\infty$ as $x \to -\infty$. Thus, the range of $\sinh : \mathbb{R} \to \mathbb{R}$ is all of $\mathbb{R}$, and $\sinh^{-1} : \mathbb{R} \to \mathbb{R}$

- Since $\sinh x$ is differentiable on $\mathbb{R}$ and $(\sinh x)' \neq 0$, the inverse function is differentiable on $\mathbb{R}$, and

$$(\sinh^{-1})'(\sinh x) = \frac{1}{(\sinh x)'} = \frac{1}{\cosh x}.$$

  Setting $y = \sinh x$, and writing

$$\cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + y^2},$$

  we conclude that

$$(\sinh^{-1})'(y) = \frac{1}{\sqrt{1 + y^2}}.$$

6. A function f has a jump discontinuity at x_0 if both the left and right limits

$$\lim_{x \to x_0^+} f(x), \qquad \lim_{x \to x_0^-} f(x)$$

exist but have different values. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable in (a, b). Prove that f' does not have a jump discontinuity in (a, b).

Solution.

- Suppose, for contradiction, that the limits

  $$L_\pm = \lim_{x \to x_0^\pm} f'(x)$$

  exist and are distinct. Let

  $$\epsilon = \frac{L_+ - L_-}{3}.$$

  Then there exists $\delta > 0$ such that

  $$|f'(x) - L_+| < \epsilon \quad \text{for } x_0 < x < x_0 + \delta$$

  and

  $$|f'(x) - L_-| < \epsilon \quad \text{for } x_0 - \delta < x < x_0.$$

  For definiteness, suppose that $L_+ > L_-$ (otherwise, we can consider $-f$ instead of f). It follows that $f'(x)$ cannot take values in the interval $[L_- + \epsilon, L_+ - \epsilon]$ of width $\epsilon > 0$ when $0 < |x - x_0| < \delta$. Since $f'(x) > L_+ - \epsilon$ for $x > x_0$ and $f'(x) < L_- + \epsilon$ for $x < x_0$, this contradicts the intermediate value property of the derivative, whatever the value of $f'(x_0)$.