**Sample Integration Questions**
**Solutions**
**Math 127B. Winter, 2005**

**1.** Give an example of a function $f : [0,1] \to \mathbb{R}$ such that $f^2$ is Riemann integrable, but $f$ is not.

**Solution.**

- The function
$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ -1 & \text{for } x \notin \mathbb{Q}, \end{cases}$$
is not Riemann integrable on $[0,1]$ since the upper Darboux sums are all equal to 1, and the lower darboux sums are all equal to $-1$. The function $f^2$ is the constant function equal to 1 which is Riemann integrable.

**2.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded, Riemann integrable function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt.$$

Prove that there exists a constant $M$ such that

$$|F(x) - F(y)| \le M|x - y| \qquad \text{for all } x, y \in [a, b].$$

Is $F$ necessarily differentiable in $(a, b)$?

**Solution.**

- Since $f$ is bounded, there is a constant $M$ such that

  $$|f(x)| \le M \qquad \text{for all } x \in [a, b].$$

  It follows that for any $x, y \in [a, b]$

  $$\begin{aligned} |F(x) - F(y)| &= \left| \int_x^y f(t)\, dt \right| \\ &\le \left| \int_x^y |f(t)|\, dt \right| \\ &\le \left| \int_x^y M\, dt \right| \\ &\le M\, |x - y|\,. \end{aligned}$$

- $F$ need not be differentiable if $f$ is not continuous. For example, if

  $$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \end{cases}$$

  then

  $$F(x) = \int_0^x f(t)\, dt = |x|$$

  is not differentiable at $x = 0$.

**3.** Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \int_0^x (x-t)g(t)\,dt.$$

Prove that $f$ satisfies the following equations:

$$f''(x) = g(x), \qquad f(0) = f'(0) = 0.$$

**Solution.**

- We can rewrite $f$ as

  $$f(x) = x\int_0^x g(t)\,dt - \int_0^x tg(t)\,dt.$$

  Differentiating this equation with respect to $x$ using the product rule and the fundamental theorem of calculus (which applies since both $g(t)$ and $tg(t)$ are continuous), we get

  $$\begin{aligned} f'(x) &= x \cdot g(x) + 1 \cdot \int_0^x g(t)\,dt - xg(x) \\ &= \int_0^x g(t)\,dt. \end{aligned}$$

  Differentiating this equation with respect to $x$ and using the fundamental theorem of calculus again, we get

  $$f''(x) = g(x).$$

- We have

  $$\begin{aligned} f(0) &= \int_0^0 (x-t)g(t)\,dt = 0, \\ f'(0) &= \int_0^0 g(t)\,dt = 0. \end{aligned}$$

**4.** Define the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

as a limit of proper integrals, and prove that it converges.

**Solution.**

- Note that
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$
so $\sin x/x$ extends to a continuous function on $[0,\infty)$. Its Riemann integral therefore exists on any bounded interval $[0,b]$ with $b > 0$. Hence we define
$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx.$$

- Let
$$I(b) = \int_0^b \frac{\sin x}{x}\, dx.$$
Then for any $b, c > 0$, we have (using an integration by parts)
$$\begin{aligned}
|I(b) - I(c)| &= \left| \int_b^c \frac{\sin x}{x}\, dx \right| \\
&= \left| \left[ \frac{-\cos x}{x} \right]_b^c - \int_b^c \frac{\cos x}{x^2}\, dx \right| \\
&\le \left| \frac{\cos b}{b} - \frac{\cos c}{c} \right| + \left| \int_b^c \frac{\cos x}{x^2}\, dx \right| \\
&\le \frac{1}{b} + \frac{1}{c} + \left| \int_b^c \frac{1}{x^2}\, dx \right| \\
&\le \frac{1}{b} + \frac{1}{c} + \left| \left[ -\frac{1}{x} \right]_b^c \right| \\
&\le \frac{1}{b} + \frac{1}{c} + \left| \frac{1}{b} - \frac{1}{c} \right| \\
&\le \frac{2}{b} + \frac{2}{c}.
\end{aligned}$$

Given any $\epsilon > 0$, let $N = 4/\epsilon$. It follows from this inequality that if $b, c > N$ then

$$|I(b) - I(c)| < \epsilon.$$

Hence, by the Cauchy criterion, the limit of $I(b)$ as $b \to \infty$ exists, and the improper integral converges.

- One can show that

  $$\int_0^\infty \frac{|\sin x|}{x}\, dx = \infty,$$

  so the integral is not absolutely convergent. This is why we had to integrate by parts (to improve the convergence of the integrals) before taking absolute values inside the integral.

- In fact, one can show using methods from complex analysis that

  $$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(This question is more difficult than the ones that will be on the final exam.)

**5.** Suppose that

$$F(x) = \begin{cases} x^2 & \text{for } 0 \le x < 2, \\ x^3 & \text{for } 2 \le x \le 3. \end{cases}$$

Evaluate the Riemann-Stieltjes integral

$$\int_0^3 x dF(x),$$

briefly justifying your computations.

**Solution.**

- Note that $F$ has a jump discontinuity of size 4 at $x = 2$. We separate $F$ into a 'jump part' $F_1$ and a 'continuous part' $F_2$ as:

$$\begin{aligned} F(x) &= F_1(x) + F_2(x), \\ F_1(x) &= \begin{cases} 0 & \text{for } 0 \le x < 2, \\ 4 & \text{for } 2 \le x \le 3, \end{cases} \\ F_2(x) &= \begin{cases} x^2 & \text{for } 0 \le x < 2, \\ x^3 - 4 & \text{for } 2 \le x \le 3. \end{cases} \end{aligned}$$

  Then, using a standard property of the Riemann-Stieltjes integral, we have

$$\int_0^3 x\,dF(x) = \int_0^3 x\,dF_1(x) + \int_0^3 x\,dF_2(x).$$

  The Riemann-Stietjes integral with respect to the jump-function $F_1$ is given by

$$\int_0^3 f(x)\,dF_1(x) = f(2) \cdot 4,$$

  so

$$\int_0^3 x\,dF_1(x) = 8.$$

  If $F$ is continuously differentiable, then

$$\int_a^b f\,dF = \int_a^b f F'\,dx$$

Hence, for the continuous part, using the standard property that we can split up the domain of integration, we have

$$\begin{aligned}
\int_0^3 x\,dF_2(x) &= \int_0^2 x\,dF_2(x) + \int_2^3 x\,dF_2(x) \\
&= \int_0^2 x\,d(x^2) + \int_2^3 x\,d(x^3-4) \\
&= \int_0^2 x\cdot 2x\,dx + \int_2^3 x\cdot 3x^2\,dx \\
&= \int_0^2 2x^2\,dx + \int_2^3 3x^3\,dx \\
&= \left[\frac{2x^3}{3}\right]_0^2 + \left[\frac{3x^4}{4}\right]_2^3 \\
&= \frac{16}{3} + \frac{243}{4} - 12 \\
&= \frac{649}{12}.
\end{aligned}$$

Hence

$$\int_0^3 x dF(x) = 8 + \frac{649}{12} = \frac{745}{12}.$$