Sample Midterm Questions
Math 127B. Winter, 2005
Closed Book. No calculators. Give complete proofs of all your answers.

1. Consider the sequence (f_n) of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \frac{nx}{\sqrt{1+n^2x^2}}.$$

Find the pointwise limit of this sequence as $n \to \infty$. Does the sequence converge uniformly on $\mathbb{R}$? Justify your answer.

2. Let

$$f_n(x) = \frac{nx + \sin(nx^2)}{n}.$$

Prove that the following limit exists, and compute its value:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx.$$

3. Prove that the following series

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2 + x^4}{n^4 + x^2}$$

converges to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

4. Determine the radius of convergence R of the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots. \end{aligned}$$

Where does the series converge? Prove that

$$f'(x) = \frac{1}{1+x^2} \qquad \text{in } |x| < R.$$

5. Suppose that (f_n) is a sequence of functions $f_n : [-1,1] \to \mathbb{R}$ that converges uniformly on $[-1,1]$ to a function $f : \mathbb{R} \to \mathbb{R}$. If the limit

$$\lim_{x\to 0} f_n(x) = a_n$$

exists for each $n \in \mathbb{N}$, and the limit

$$\lim_{n\to\infty} a_n = a$$

exists, prove that $\lim_{x\to 0} f(x)$ exists, and

$$\lim_{x\to 0} f(x) = a$$

Give a counter-example to show that this result need not be true if (f_n) converges to f pointwise, but not uniformly.