The Chain Rule

**Definition 1.** If $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ where $f(U) \subset V$, then the composition $g \circ f : U \to \mathbb{R}$ of $f$ and $g$ is given by $(g \circ f)(x) = g(f(x))$.

**Theorem 2.** Suppose that $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ where $U, V \subset \mathbb{R}$ are open sets and $f(U) \subset V$. If $f$ is differentiable at $c \in U$ and $g$ is differentiable at $f(c) \in V$, then $g \circ f$ is differentiable at $c$ and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

**Proof.** Define $D : V \to \mathbb{R}$ by

$$D(y) = \frac{g(y) - g(f(c))}{y - f(c)} \quad \text{if} \quad y \neq f(c), \quad D(f(c)) = g'(f(c)).$$

Then $\lim_{y \to f(c)} D(y) = D(f(c))$, since $g$ is differentiable at $f(c)$, so $D$ is continuous at $f(c)$.

For every $y \in V$, we have that

$$g(y) - g(f(c)) = D(y) \cdot [y - f(c)],$$

since we can cancel $y - f(c) \neq 0$ on the right-hand side if $y \neq f(c)$, and both sides are equal to 0 if $y = f(c)$. It follows that for any $x \in U$ with $x \neq c$, we have

$$g(f(x)) - g(f(c)) = D(f(x)) \cdot \frac{f(x) - f(c)}{x - c}. \quad (1)$$

We note that $f$ is continuous at $c$, since $f$ is differentiable at $c$, and $D$ is continuous at $f(c)$, so $D \circ f : U \to \mathbb{R}$ is continuous at $c$, since the composition of continuous functions is continuous.

Taking the limit of (1) as $x \to c$ and using the algebraic properties of limits, the continuity of $D \circ f$, and the differentiability of $f$, we get that

$$\lim_{x \to c} \left[ \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} \right] = \lim_{x \to c} \left[ (D \circ f)(x) \cdot \frac{f(x) - f(c)}{x - c} \right]$$

$$= \lim_{x \to c} [(D \circ f)(x)] \cdot \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \right]$$

$$= (D \circ f)(c) \cdot f'(c)$$

$$= g'(f(c)) \cdot f'(c),$$

which proves that $g \circ f$ is differentiable at $c$ with the given derivative. \qed