1. Define \( f_n : [0, \infty) \to \mathbb{R} \) by
\[
f_n(x) = \frac{1}{1 + x^n}.
\]
(a) Find the pointwise limit of \((f_n)\) on \([0, \infty)\).
(b) Does the sequence converge uniformly on \([0, \infty)\)? Justify your answer.

2. (a) Find all points \( x \in \mathbb{R} \) where the following power series converges:
\[
f(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{n2^n} x^n.
\]
(b) Show that there exists a constant \( C \) such that
\[
f(x) = \ln(2 + x) + C \quad \text{for} \ |x| < R,
\]
where \( R \) is the radius of convergence of the power series for \( f \) in (a).

3. (a) Define what it means for a function \( f : \mathbb{R} \to \mathbb{R} \) to be differentiable at \( c \in \mathbb{R} \).
(b) If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at \( c \), prove that
\[
f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c - h)}{2h}.
\]

4. Let \( a > 0 \). Give a definition of the following improper Riemann integral as a limit of Riemann integrals:
\[
\int_{2}^{\infty} \frac{1}{x(\log x)^a} \, dx.
\]
For what values of \( a \) does this integral converge?
5. In each case, give an example of a function \( f : [0, 1] \to \mathbb{R} \) with the stated property, or explain why such a function doesn’t exist.

(a) \( f \) is unbounded and Riemann integrable.
(b) \( f \) is bounded and not Riemann integrable.
(c) \( f \) is discontinuous and Riemann integrable.
(d) \( f \) is continuous and not Riemann integrable.

6. Suppose that \( f : [0, \pi] \to \mathbb{R} \) is a continuously differentiable function. Prove that
\[
\lim_{n \to \infty} \int_0^\pi f(x) \sin(nx) \, dx = 0.
\]
HINT. Integrate by parts.

7. (a) Find the Taylor series of \( e^{-x} \) (at \( x = 0 \)).
(b) Give an expression for the remainder \( R_n(x) \) between \( e^{-x} \) and its Taylor polynomial of degree \( n - 1 \) involving an intermediate point \( \xi \) between 0 and \( x \).
(c) Prove from your expression in (b) that the Taylor series for \( e^{-x} \) converges to \( e^{-x} \) for every \( x \in \mathbb{R} \).

8. Define \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
  x^2 [\sin(1/x) - 2] & \text{for } x \neq 0, \\
  0 & \text{for } x = 0.
\end{cases}
\]

(a) Prove that \( f(x) \) has a strict maximum at \( x = 0 \) (i.e. \( f(0) > f(x) \) for all \( x \neq 0 \)).
(b) Prove that \( f \) is differentiable on \( \mathbb{R} \). What is \( f'(0) \)?
(c) Prove that \( f \) is not increasing on the interval \((-\epsilon, 0)\) and \( f \) is not decreasing on the interval \((0, \epsilon)\) for any \( \epsilon > 0 \).