1. Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.
   (a) If the series $\sum_{n=0}^{\infty} a_n (x - 1)^n$ converges absolutely for $x = 2$, then the series converges uniformly on $[0, 2]$.
   (b) If $f : [0, 1] \to (0, \infty)$ is Riemann integrable and $g = 1/f$, then $g : [0, 1] \to (0, \infty)$ is Riemann integrable.
   (c) If $f : [0, 1] \to \mathbb{R}$ is Riemann integrable and $F(x) = \int_0^x f$, then $F : [0, 1] \to \mathbb{R}$ is differentiable.
   (d) If $(f_n)$ is a sequence of Riemann integrable functions $f_n : [0, \pi/2] \to \mathbb{R}$ that converges uniformly to $\cos x$ on $[0, \pi/2]$, then $\int_0^{\pi/2} f_n \to 1$ as $n \to \infty$.

Solution

• (a) True. By assumption, $\sum |a_n|$ converges. Since $|a_n(x - 1)^n| \leq |a_n|$ for $x \in [0, 2]$, the M-test implies that the series converges uniformly on $[0, 2]$.

• (b) False. For example,

$$f(x) = \begin{cases} 
  x & \text{if } 0 < x \leq 1 \\
  1 & \text{if } x = 0
\end{cases}$$

is integrable of $[0, 1]$, since it differs from the continuous and therefore integrable function $h(x) = x$ at one point, but $1/f$ is unbounded on $[0, 1]$, so it isn’t Riemann integrable.

• (c) False. For example, if

$$f(x) = \begin{cases} 
  0 & \text{if } 0 \leq x \leq 1/2 \\
  1 & \text{if } 1/2 < x \leq 1
\end{cases} \quad F(x) = \begin{cases} 
  0 & \text{if } 0 \leq x \leq 1/2 \\
  x - 1/2 & \text{if } 1/2 < x \leq 1
\end{cases}$$

then $F$ isn’t differentiable at $1/2$. (The result is true if $f$ is continuous.)

• True. The uniform convergence of the $f_n$ implies that

$$\int_0^{\pi/2} f_n \to \int_0^{\pi/2} \cos x = [\sin x]_0^{\pi/2} = 1.$$
2. Find all points $x \in \mathbb{R}$ where the following power series converges:

$$
\sum_{n=0}^{\infty} \frac{1}{1 + n2^n} x^n.
$$

**Solution**

- Let $a_n = 1/(1 + n2^n)$ denote the $n$th coefficient of the power series. Then

  $$
  \left| \frac{a_{n+1}}{a_n} \right| = \frac{1 + n2^n}{1 + (n + 1)2^{n+1}} = \frac{2^{-n}/n + 1}{2^{-n}/n + (1 + 1/n) \cdot 2} \to \frac{1}{2}.
  $$

  The ratio test implies that the radius of convergence is $R = 1/(1/2) = 2$, so the series converges absolutely in $|x| < 2$ and diverges in $|x| > 2$.

- If $x = 2$, then the series becomes

  $$
  \sum_{n=0}^{\infty} c_n, \quad c_n = \frac{2^n}{1 + n2^n} = \frac{1}{2^{-n} + n} \geq \frac{1}{n+1},
  $$

  so the series diverges by comparison with the divergent harmonic series.

- If $x = -2$, then the series becomes

  $$
  \sum_{n=0}^{\infty} (-1)^n c_n, \quad c_n = \frac{1}{2^{-n} + n}.
  $$

  The sequence $n + 2^{-n}$ is increasing, since

  $$
  n + 1 + 2^{-(n+1)} > n + 1 \geq n + 2^{-n},
  $$

  so $c_n \to 0$ is decreasing, and the series converges conditionally by the alternating series test.
3. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is $f$ Riemann integrable on $[0, 1]$? Prove your answer.

**Solution**

- For every partition $P$, we have $L(f, P) = 0$, since every interval $I_k$ with $|I_k| > 0$ contains irrational numbers and $\inf_{I_k} f = 0$, so the lower integral of $f$ is $L(f) = 0$.

- The upper sums $U(f, P)$ of $f$ are the same as the upper sums $U(g, P)$ of the integrable function $g(x) = x$, since every interval contains rational points where $f(x) = x$. It follows that

$$U(f) = U(g) = \int_0^1 x \, dx = \frac{1}{2},$$

so $L(f) \neq U(f)$, and $f$ isn’t Riemann integrable.
4. Prove that

\[ 0 \leq \int_{0}^{\pi/2} \sin(\sin x) \, dx \leq 1. \]

(You can use any standard properties of the sine function.)

Solution

• Since \( 0 \leq \sin x \leq 1 \) on \([0, \pi/2]\) and \( \sin \theta \geq 0 \) for \( 0 \leq \theta \leq 1 \), we have that \( \sin(\sin x) \geq 0 \) on \([0, \pi/2]\), so

\[ \int_{0}^{\pi/2} \sin(\sin x) \, dx \geq 0 \]

by the monotonicity of the integral. (Note that \( \sin(\sin x) \) is a composition of continuous functions, so it’s continuous and therefore integrable.)

• On \([0, \pi/2]\), the function \( \sin x \) is monotonic increasing, since \( (\sin x)^{'} = \cos x \geq 0 \), and \( \sin x \leq x \), since \( \sin x - x = 0 \) at \( x = 0 \) and \( \sin x - x \) is decreasing since \( (\sin x - x)^{'} = \cos x - 1 \leq 0 \). It follows that \( \sin(\sin x) \leq \sin x \) on \([0, \pi/2]\), so

\[ \int_{0}^{\pi/2} \sin(\sin x) \, dx \leq \int_{0}^{\pi/2} \sin x \, dx = 1. \]
5. (a) State a theorem which guarantees that the series \( \sum_{n=1}^{\infty} f_n(x) \) is differentiable with derivative \( \sum_{n=1}^{\infty} f'_n(x) \).

(b) Prove that the following series converge uniformly on \( \mathbb{R} \):

\[
 f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^3}, \quad g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}.
\]

(c) Are \( f, g \) continuous on \( \mathbb{R} \)?

(d) What can you say about the differentiability of \( f, g \)?

Solution

- (a) Suppose that \( f_n : (a, b) \to \mathbb{R} \) is differentiable for each \( n \in \mathbb{N} \), \( \sum_{n=1}^{\infty} f_n \) converges pointwise to \( f \) on \( (a, b) \), and \( \sum_{n=1}^{\infty} f'_n \) converges uniformly to \( g \) on \( (a, b) \). Then \( f : (a, b) \to \mathbb{R} \) is differentiable on \( (a, b) \) and \( f' = g \).

- In fact, it’s enough to assume that the series \( \sum_{n=1}^{\infty} f_n(c) \) converges at one point \( c \in (a, b) \), in which case the uniform convergence of \( \sum_{n=1}^{\infty} f'_n \) implies that \( \sum_{n=1}^{\infty} f_n \) converges uniformly on \( (a, b) \) to a function \( f \).

- (b) Since

\[
 \left| \frac{\cos nx}{n^3} \right| \leq \frac{1}{n^3}, \quad \left| \frac{\sin nx}{n^2} \right| \leq \frac{1}{n^2},
\]

and \( \sum 1/n^3, \sum 1/n^2 \) converge, the M-test implies that the series for \( f, g \) converge uniformly on \( \mathbb{R} \).

- (c) The functions \( \cos nx, \sin nx \) are continuous of \( \mathbb{R} \), so the uniform convergence of the series implies that \( f, g \) are continuous on \( \mathbb{R} \).

- (d) Since the series for \( f, g \) converge uniformly, the theorem in (a) implies that \( f \) is differentiable on \( \mathbb{R} \) with

\[
 f'(x) = \sum_{n=1}^{\infty} \frac{(\cos nx)}{n^3}' = -\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} = -g(x).
\]

- On the other hand, the term-by-term derivative of the series for \( g \),

\[
 \sum_{n=1}^{\infty} \frac{(\sin nx)}{n^2}' = \sum_{n=1}^{\infty} \frac{\cos nx}{n},
\]
doesn’t converge uniformly or even pointwise on \( \mathbb{R} \); for example, at \( x = 0 \) it is the divergent harmonic series. So we can’t conclude anything about the differentiability of \( g \) from the theorem in (a).

- In fact, as the divergence of the formal series for \( g'(0) \) suggests, \( g \) isn’t differentiable at 0, but this requires a separate proof (see optional note below).

**Proposition 1.** The function \( g : \mathbb{R} \rightarrow \mathbb{R} \) is not differentiable at 0.

**Proof.** The derivative exists if and only if the following limit exists:

\[
g'(0) = \lim_{x \to 0} \left( \frac{g(x) - g(0)}{x} \right) = \lim_{x \to 0} h(x), \quad h(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 x}.
\]

Since \( \frac{\sin \theta}{\theta} \to 1 \) as \( \theta \to 0 \), we have formally that

\[
\lim_{x \to 0} h(x) = \lim_{x \to 0} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 x} = \sum_{n=1}^{\infty} \frac{1}{n} \lim_{x \to 0} \frac{\sin nx}{nx} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,
\]

but this exchange of limits requires justification.

Consider the sequence \( x_N = \frac{1}{N} \) with \( N \in \mathbb{N} \), so \( x_N \to 0 \) as \( N \to \infty \). Then

\[
h(x_N) = \sum_{n=1}^{N} \frac{\sin(nx_N)}{n^2 x_N} + \sum_{n=N+1}^{\infty} \frac{\sin(nx_N)}{n^2 x_N}.
\]

For \( 0 \leq \theta \leq 1 \), we have \( \sin \theta \geq C \theta \), where \( C = \sin 1 \), so

\[
\sum_{n=1}^{N} \frac{\sin(n x_N)}{n^2 x_N} \geq C \sum_{n=1}^{N} \frac{1}{n} \to \infty \quad \text{as } N \to \infty.
\]

Also, comparison of an improper Riemann integral with its lower sums shows that

\[
\left| \sum_{n=N+1}^{\infty} \frac{\sin(nx_N)}{n^2 x_N} \right| \leq N \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq N \int_{N}^{\infty} \frac{1}{t^2} dt = 1.
\]

Taking the limit of (1) as \( N \to \infty \), we see that \( h(x_N) \to \infty \), so \( \lim_{x \to 0} h(x) \) does not exist. \( \square \)