5.2.5 (a) Continuous for $a > 0$.
(b) Differentiable for $a > 1$ with 

$$f'_a(x) = \begin{cases} \frac{ax}{a-1} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The derivative is continuous at zero.
(c) Twice differentiable if $a > 2$.

5.2.6 (a) The limit $x \to c$ is equivalent to the limit $h \to 0$ with $x = c + h$.
(b) If $g$ is differentiable at $c$, then (writing $h = -k$)

$$\lim_{h \to 0} \frac{g(c + h) - g(c - h)}{2h} = \lim_{h \to 0} \frac{g(c + h) - g(c) + g(c) - g(c - h)}{2h} = \lim_{h \to 0} \frac{g(c + h) - g(c)}{2h} + \lim_{k \to 0} \frac{g(c + k) - g(c)}{2k} = g'(c).$$

5.2.7 (a) $1 < a < 2$. (b) $2 < a \leq 3$. (c) $3 < a < 4$.

5.2.9 (a) True. By Darboux’s theorem, $f'$ takes on all values between any two distinct values, which includes irrational values.
(b) False. For example, the function $g(x)$ in 5.2.10 has $g'(0) = 1/2 > 0$ but there are points in any $\delta$-neighborhood of 0 with $g'(x) < 0$.
(c) True. By the mean value theorem, there exists $c$ between 0 and $x$ such that

$$\frac{f(x) - f(0)}{x} = f'(c).$$

Then $c \to 0$ as $x \to 0$, so

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{c \to 0} f'(c) = L.$$

5.2.10 See Example 8.38 in the lecture notes.
5.2.12 By the definition of the inverse function, we have 
\[ f^{-1}(f(x)) = x. \]
The chain rule applies under the given hypotheses and implies that 
\[ (f^{-1})'(f(x)) f'(x) = 1. \]

5.3.1 (a) If \( f' \) is continuous on the compact interval \([a, b]\), then \( f' \) is bounded by the Weierstrass extreme value theorem, so there exists \( M \geq 0 \) such that 
\[ |f'(x)| \leq M \text{ for all } a \leq x \leq b. \]
By the mean value theorem for any distinct \( x, y \in [a, b] \), there exists \( c \) between \( x \) and \( y \) such that 
\[ \left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq M, \]
meaning that \( f \) is Lipschitz on \([a, b]\).

(b) By the Weierstrass extreme value theorem, \( |f'| \) attains its maximum value \( M \geq 0 \) on \([a, b]\), and since \( |f'(x)| < 1 \) for every \( x \in [a, b] \), we have \( M < 1 \). It follows from (a) that 
\[ |f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in [a, b], \]
meaning that \( f \) is a contraction on \([a, b]\).

5.3.2 Since \( f' \) is never zero, the Darboux theorem implies that \( f' \) cannot change sign on \( A \): If \( f'(x) < 0 \) and \( f'(y) > 0 \), then \( f'(c) = 0 \) for some \( c \) between \( x \) and \( y \). If \( f' > 0 \) (or \( f' < 0 \)) on \( A \), then \( f \) is strictly increasing (or strictly decreasing), so \( f \) is one-to-one. The function \( f(x) = x^3 \) is one-to-one on \( \mathbb{R} \), but \( f'(0) = 0 \).

5.3.4 (a) Since \( f \) is differentiable at 0, it is continuous at 0, so by the sequential characterization of continuity 
\[ f(0) = \lim_{n \to \infty} f(x_n) = 0. \]
Similarly, for the derivative, the sequential characterization of limits gives 
\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{n \to \infty} \frac{f(x_n)}{x_n} = 0. \]

(b) We can extract from \((x_n)\) a strictly monotone subsequence, still denoted by \((x_n)\), that converges to 0. Suppose for definiteness that \((x_n)\) is strictly decreasing. By the mean value theorem, for every \( n \in \mathbb{N} \) there exists \( x_{n+1} < c_n < x_n \) such that 
\[ f'(c_n) = \frac{f(x_n) - f(x_{n+1})}{x_n - x_{n+1}} = 0. \]
The argument in (a) applied to \( f' \) with the sequence \((c_n)\) then implies that 
\[ f'(0) = f''(0) = 0. \]