Math 127B, Spring 2019  
Set 5: Solutions

6.4.2 (a) True. \( S_n = \sum_{k=1}^{n} g_k \) converges uniformly to the sum \( S \) and the difference of two uniformly convergent sequences converges uniformly, so \( g_n = S_n - S_{n-1} \) converges uniformly to \( S - S = 0 \).

(b) True. Since

\[ 0 \leq \sum_{k=n+1}^{m} f_k \leq \sum_{k=n+1}^{m} g_k, \]

the series \( \sum f_n \) satisfies the Cauchy condition for uniform convergence.

(c) False. For example, let \( f_n : \mathbb{R} \to \mathbb{R} \) be the constant function

\[ f_n(x) = \frac{(-1)^{n+1}}{n}. \]

Then \( \sum f_n \) converges uniformly on \( \mathbb{R} \), since \( \sum (-1)^{n+1}/n \) converges, but \( |f_n| = 1/n \) and \( \sum 1/n \) diverges.

6.4.4 If \( |x| < 1 \), then

\[ 0 \leq \frac{x^{2n}}{1 + x^{2n}} \leq x^{2n}, \]

so the series converges pointwise by comparison with a convergent geometric series. The M-test implies that the convergence is uniform on \( |x| \leq r \) for every \( 0 < r < 1 \), so the sum is continuous on \([-r, r]\), and hence on

\[ (-1, 1) = \bigcup_{0 < r < 1} [-r, r]. \]

If \( |x| \geq 1 \), then

\[ \lim_{n \to \infty} \frac{x^{2n}}{1 + x^{2n}} \neq 0, \]

so the series diverges pointwise.

6.4.10 Since \( |u_n(x)| \leq 1/2^n \) for every \( x \in \mathbb{R} \) and \( \sum 1/2^n \) converges, the M-test implies that \( \sum u_n \) converges uniformly on \( \mathbb{R} \). Each \( u_n \) is continuous at every irrational number, so the sum is continuous at every irrational number, since uniform convergence preserves continuity. If \( x < y \), then

\[ h(x) = \sum_{r_n < x} \frac{1}{2^n} < \sum_{r_n < y} \frac{1}{2^n} = h(y), \]

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so $h : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing, with $h(x) \to 0$ as $x \to -\infty$ and $h(x) \to 1$ as $x \to \infty$.

6.5.2 (a) The power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

converges for every $x \in \mathbb{R}$.

(b) Not possible, since the power series always converges when $x = 0$ and all terms with $n \geq 1$ are zero.

(c) The power series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$$

has radius of convergence $R = 1$ and converges absolutely for $x = \pm 1$.

6.5.5 See Theorem 10.20 in the lecture notes.

6.5.7 See Theorems 10.5–10.6 in the lecture notes.