7.4.1 (a) The reverse triangle inequality gives
\[ ||f(x)| - |f(y)|| \leq |f(x) - f(y)| \quad \text{for all } x, y \in A,\]
so (see Proposition 11.8 in the lecture notes)
\[ \sup |f(x)| - \inf |f(x)| \leq \sup f - \inf f.\]
In particular, the difference between the upper and lower sums of \(|f|\) is bounded by the difference between the upper and lower sums of \(f\).
(b) It follows from (a) and the integrability of \(f\) that for any \(\epsilon > 0\), there exists a partition \(P\) of \([a,b]\) such that
\[ U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) < \epsilon, \]
so the Cauchy criterion for integrability implies that \(|f|\) is integrable.
(c) Since \(-|f| \leq f \leq |f|\), we have \(-\int_{a}^{b} |f| \leq \int_{a}^{b} f \leq \int_{a}^{b} |f|\) by monotonicity of the integral.

7.4.3 (a) False. For example if
\[ f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \cap [a, b], \\
-1 & \text{if } x \in \mathbb{Q}^c \cap [a, b],
\end{cases} \]
then \(f\) isn’t integrable, but \(|f| = 1\) is integrable.
(b) False. For example, the function \(f : [0, 1] \to \mathbb{R}\) defined in Problem 7.3.3,
\[ f(x) = \begin{cases} 
1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\
0 & \text{otherwise},
\end{cases} \]
it strictly positive at infinitely many points, but its integral is zero.
(c) True. If \(g\) is continuous at \(y_0 \in [a, b]\) with \(g(y_0) = 2\epsilon > 0\), and assume for definiteness that \(a < y_0 < b\) (a similar argument applies if \(y_0\) is an endpoint). Then there exists \(\delta > 0\) such that \(g(x) \geq \epsilon\) for \(|x - y_0| \leq \delta\), and, since \(g \geq 0\) on \([a, b]\), we have
\[ \int_{a}^{b} g = \int_{a}^{y_0 - \delta} g + \int_{y_0 - \delta}^{y_0 + \delta} g + \int_{y_0 + \delta}^{b} g \geq 0 + 2\epsilon \delta + 0 > 0. \]
7.5.1 (a) We have

\[
F(x) = \begin{cases} 
(1 + x^2)/2 & \text{if } x \geq 0, \\
(1 - x^2)/2 & \text{if } x < 0.
\end{cases} = \frac{1}{2}(1 + x|x|)
\]

Then \(F\) is continuous and differentiable on \(\mathbb{R}\). This is clear for \(x \neq 0\), and

\[
F'(0) = \lim_{x \to 0} \frac{F(x) - F(0)}{x} = \frac{1}{2} \lim_{x \to 0} |x| = 0.
\]

(b) In this case,

\[
F(x) = \begin{cases} 
1 + 2x & \text{if } x \geq 0, \\
1 + x & \text{if } x < 0.
\end{cases}
\]

Then \(F\) is continuous on \(\mathbb{R}\) and differentiable on \(\mathbb{R} \setminus \{0\}\). It isn’t differentiable at \(x = 0\), since its left and right derivatives are different:

\[
F'(0^+) = \lim_{x \to 0^+} \frac{F(x) - F(0)}{x} = 2, \quad F'(0^-) = \lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = 1.
\]

7.5.4 Since \(f\) is continuous, the fundamental theorem of calculus implies that \(F(x) = \int_a^x f\) is differentiable and \(F' = f\). Since \(F = 0\), we have \(f = 0\). On the other hand, if

\[
f(x) = \begin{cases} 
0 & \text{if } a \leq x < b \\
1 & \text{if } x = b
\end{cases}
\]

then \(\int_a^x f = 0\) for every \(a \leq x \leq b\), but \(f \neq 0\). (Or, for a more complicated counter-example, consider the Thomae function in Exercise 7.3.2.)

7.5.8 (a) We have \(L(1) = 0\) and \(L'(x) = 1/x\) for \(x > 0\). The only function with these properties is \(L(x) = \ln x\).

(b) By the chain rule,

\[
\partial_x L(xy) = L'(xy) \cdot y = \frac{1}{xy} \cdot y = \frac{1}{x}.
\]

It follows that

\[
\partial_x [L(xy) - L(x)] = 0,
\]

so \(L(xy) - L(x) = C(y)\) for some function of integration \(C(y)\). Evaluating this equation at \(x = 1\), we get that \(C(y) = L(y)\), so \(L(xy) = L(x) + L(y)\).
(c) Setting \( y = 1/x \) and using the fact that \( L(1) = 0 \), we have that
\[
0 = L(x) + L\left(\frac{1}{x}\right),
\]
so \( L(1/x) = -L(x) \). Then for all \( x, y > 0 \)
\[
L\left(\frac{x}{y}\right) = L\left(x \cdot \frac{1}{y}\right) = L(x) + L\left(\frac{1}{y}\right) = L(x) - L(y).
\]

(d) By the additivity and monotonicity of the integral, we have
\[
L(n) = \sum_{k=1}^{n-1} \int_{k}^{k+1} \frac{1}{x} \, dx, \quad \frac{1}{k+1} \leq \int_{k}^{k+1} \frac{1}{x} \, dx \leq \frac{1}{k}.
\]
It follows that
\[
\gamma_n - \frac{1}{n} = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \int_{k}^{k+1} \frac{1}{x} \, dx\right).
\]
We see that \( \gamma_n - 1/n \) is increasing in \( n \), since the terms in the sum are always positive, and bounded from above, since
\[
\gamma_n - \frac{1}{n} \leq \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) \leq 1.
\]
Hence, by the monotone convergence theorem for sequences, \( \gamma_n - 1/n \to \gamma \) as \( n \to \infty \) for some \( 0 \leq \gamma \leq 1 \), so we also have \( \gamma_n \to \gamma \).

(e) We have
\[
\gamma_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - L(2n)
= 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(2n),
\]
so, using the fact that \( L(2n) - L(n) = L(2) \), we get that
\[
\gamma_{2n} - \gamma_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(2)
= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} - L(2).
\]
Since \( \gamma_n \to \gamma \) and \( \gamma_{2n} \to \gamma \) as \( n \to \infty \), we obtain that the sum of the alternating harmonic series is given by
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n n+1}{n} + \cdots = \ln 2.
\]

7.5.10 See Theorem 12.12 in the lecture notes.