Math 127C, Spring 2006
Final Exam Solutions

1. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x_1, x_2) = (\sin x_2 - x_1, e^{x_1} - x_2),$$
$$g(y_1, y_2) = (y_1 y_2, y_1^2 + y_2^2).$$

Use the chain rule to compute the matrix of $(g \circ f)'(0, 0)$.

**Solution.**

- By the chain rule,

$$(g \circ f)'(0) = g'(f(0))f'(0).$$

- The Jacobian matrices of $f'$ and $g'$ are

$$[f'(x_1, x_2)] = \begin{pmatrix}-1 & \cos x_2 \\ e^{x_1} & -1\end{pmatrix}, \quad [g'(y_1, y_2)] = \begin{pmatrix}y_2 & y_1 \\ 2y_1 & 2y_2\end{pmatrix},$$

and $f(0, 0) = (0, 1)$.

- Thus,

$$[f'(0, 0)] = \begin{pmatrix}-1 & 1 \\ 1 & -1\end{pmatrix}, \quad [g'(0, 1)] = \begin{pmatrix}1 & 0 \\ 0 & 2\end{pmatrix},$$

and

$$[(g \circ f)'(0, 0)] = \begin{pmatrix}1 & 0 \\ 0 & 2\end{pmatrix} \begin{pmatrix}1 & 0 \\ -1 & 1\end{pmatrix} \begin{pmatrix}-1 & 1 \\ 1 & -1\end{pmatrix} \begin{pmatrix}1 \\ 2\end{pmatrix}.$$
2. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (x^2y + x, 6x + y^2).$$

What does the inverse function theorem imply about the existence of a local inverse of $f$ at: (a) $(x, y) = (1, 1)$; (b) $(x, y) = (-1, 1)$?

Solution.

- The partial derivatives of the components of $f$ exist and are continuous in $\mathbb{R}^2$, so $f$ is continuously differentiable in $\mathbb{R}^2$, and the inverse function theorem applies at points where the derivative of $f$ is invertible.

- The Jacobian matrix of $f'$ is

$$[f'(x, y)] = \begin{pmatrix} 2xy + 1 & x^2 \\ 6 & 2y \end{pmatrix}.$$

- (a) At $(1, 1)$ we have

$$[f'(1, 1)] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$  

The determinant of this matrix is zero, so $f'(1, 1)$ is singular and there is no conclusion from the inverse function theorem.

- (b) At $(-1, 1)$ we have

$$[f'(-1, 1)] = \begin{pmatrix} -1 & 1 \\ 6 & 2 \end{pmatrix}.$$  

The determinant of this matrix is non-zero, so $f'(-1, 1)$ is nonsingular. The inverse function theorem implies that there are open sets $U, V \subset \mathbb{R}^2$ with $(-1, 1) \in U$ and $(0, -5) \in V$ such that $f : U \to V$ is one-to-one and onto. Moreover, the local inverse $f^{-1} : V \to U$ is continuously differentiable.
3. Let $I = [0, 1] \times [0, 1]$, and define $f : I \to \mathbb{R}$ by

$$f(x, y) = y \sin(\pi xy).$$

Why is $f$ Riemann integrable over $I$? Evaluate $\int_I f \, dx \, dy$, and justify your calculations.

**Solution.**

- The function is continuous on $I$, since it is the product and composition of continuous functions. Therefore it is Riemann integrable on $I$.

- Since $f$ is continuous, the Riemann integral on $I$ is equal to the iterated integrals by Fubini’s theorem. Performing the $x$-integral first followed by the $y$-integral, and using the fundamental theorem of calculus, we get

$$\int_I f \, dx \, dy = \int_0^1 \left( \int_0^1 y \sin(\pi xy) \, dx \right) \, dy$$

$$= \int_0^1 \left[ -\frac{1}{\pi} \cos(\pi xy) \right]_{x=0}^{x=1} \, dy$$

$$= \frac{1}{\pi} \int_0^1 [1 - \cos(\pi y)] \, dy$$

$$= \frac{1}{\pi} \left[ y - \frac{1}{\pi} \sin(\pi y) \right]_{y=0}^{y=1}$$

$$= \frac{1}{\pi}.$$
4. Let 
\[ \omega = f dx + dy, \quad \lambda = g dx + dz \]
be one-forms in \( \mathbb{R}^3 \), where \( f(x, y, z) \) and \( g(x, y, z) \) are smooth functions.
(a) Calculate \( \omega \wedge \lambda \) and express your answer in standard form.
(b) Calculate \( d(\omega \wedge \lambda) \) and express your answer in standard form.

**Solution.**

- (a) Using the anti-symmetry of the wedge product, we compute that
  \[
  \omega \wedge \lambda = (fdx + dy) \wedge (gdx + dz) \\
  = gdy \wedge dx + f dx \wedge dz + dy \wedge dz \\
  = -gdx \wedge dy + f dx \wedge dz + dy \wedge dz.
  \]

- (b) From (a), we compute that
  \[
  d(\omega \wedge \lambda) = d(-gdx \wedge dy + f dx \wedge dz + dy \wedge dz) \\
  = -g_z dz \wedge dx \wedge dy + f_y dy \wedge dx \wedge dz \\
  = - (f_y + g_z) dx \wedge dy \wedge dz.
  \]

5. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
  \[
  f(x, y) = \begin{cases} 
  x^2 \sin(1/y) & \text{if } y \neq 0 \\
  0 & \text{if } y = 0
  \end{cases}
  \]
Determine, with proof, at what points of \( \mathbb{R}^2 \) the function \( f \) is differentiable.

**Solution.**

- By the chain, product, and quotient rules, the partial derivatives of \( f \) exist and are continuous in the open set \( E = \{(x, y) : y \neq 0\} \), so \( f \) is continuously differentiable in \( E \).
- If \( x \neq 0 \) then \( \lim_{y \to 0} f(x, y) \) does not exist, so \( f \) is not continuous at \( (x, 0) \), and therefore \( f \) is not differentiable.
• We claim that \( f \) is differentiable at \((0,0)\) with \( f'(0,0) = 0 \). To prove this, note that for \( h = (h, k) \in \mathbb{R}^2 \) with \( k \neq 0 \), we have
\[
|f(0 + h, 0 + k) - f(0, 0)| = \left| h^2 \sin \frac{1}{k} - 0 \right| \leq h^2 \leq |h|^2,
\]
while if \( k = 0 \), we have
\[
|f(0 + h, 0 + k) - f(0, 0)| = 0.
\]
It follows that
\[
\lim_{h \to 0} \frac{|f(0 + h) - f(0)|}{|h|} = 0,
\]
which is what we had to prove.

• Thus, \( f \) is differentiable at all points \((x, y)\) \(\in \mathbb{R}^2\) except those with \( y = 0 \) and \( x \neq 0 \).

6. Let \( I = [0,1] \times [0,1] \) and define \( f : I \to \mathbb{R} \) by
\[
f(x, y) = \begin{cases} 
1 & \text{if } x = y, \\
0 & \text{if } x \neq y.
\end{cases}
\]
Prove that \( f \) is Riemann integrable on \( I \), and evaluate \( \int_I f \, dx \, dy \).

Solution.

• For \( N \in \mathbb{N} \), consider a partition \( \mathcal{P}_N \) of \( I \) into \( N^2 \) equal closed rectangles obtained by partitioning each side \([0,1]\) into \( N \) equal intervals.

• The infimum of \( f \) on every rectangle is 0, so the lower Riemann sum of \( f \) associated with the partition \( \mathcal{P}_N \) is
\[
\mathcal{L}(f; \mathcal{P}_N) = 0.
\]

• The supremum of \( f \) on a rectangle in the partition is equal to 0 except for those rectangles that intersect the diagonal \( x = y \), where the supremum of \( f \) is equal to 1. There are \( 3N - 2 \) such rectangles (\( N \) on the diagonal and \( 2(N - 1) \) adjacent to the diagonal). The area of each rectangle is \( 1/N^2 \), so it follows that the upper Riemann sum of \( f \) associated with \( \mathcal{P} \) is given by
\[
\mathcal{U}(f; \mathcal{P}_N) = 1 \cdot (3N - 2) \cdot \frac{1}{N^2}
\]

5
• The right-hand side of this equation tends to 0 as $N \to \infty$. It then follows from standard properties of Riemann sums that

$$0 = \sup_{N \in \mathbb{N}} \mathcal{L}(f; \mathcal{P}_N) \leq \int_I f \, dx \, dy \leq \inf_{N \in \mathbb{N}} \mathcal{U}(f; \mathcal{P}_N) = 0.$$ 

Hence

$$\int_I f \, dx \, dy = \int_I f \, dx \, dy = 0,$$

so $f$ is Riemann integrable on $I$, and

$$\int_I f \, dx \, dy = 0.$$

7. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers, and define

$$d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$$

by

$$d_1(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|, \quad d_2(n, m) = |n - m|.$$

(a) Prove that $d_1$, $d_2$ are metrics on $\mathbb{N}$.

(b) Determine whether or not $\mathbb{N}$ is complete with respect each of the metrics $d_1$, $d_2$.

Solution.

• (a) Both $d_1$, $d_2$ are symmetric, non-negative, and equal to zero if and only if $n = m$. For $d_1$, we have

$$d_1(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{1}{n} - \frac{1}{p} + \frac{1}{p} - \frac{1}{m}\right| \leq \left|\frac{1}{n} - \frac{1}{p}\right| + \left|\frac{1}{p} - \frac{1}{m}\right| \leq d_1(n, p) + d_2(p, m).$$

The triangle inequality for $d_2$ is immediate.
• (b) The metric space \((\mathbb{N}, d_1)\) is not complete.

• For example, consider the sequence \(\{x_k\}\) with \(x_k = k\). This is a Cauchy sequence with respect to \(d_1\): Given \(\epsilon > 0\), choose \(N = 2/\epsilon\). Then \(j, k > N\) implies that

\[
d_1(x_j, x_k) = \left| \frac{1}{j} - \frac{1}{k} \right| \\
\leq \frac{1}{j} + \frac{1}{k} \\
< \frac{2}{N} \\
< \epsilon.
\]

• If \(x \in \mathbb{N}\) is any integer, choose \(\epsilon = 1/(2x) > 0\). Then for all \(k > 2x\) we have

\[
d(x_k, x) = \frac{1}{x} - \frac{1}{k} > \frac{1}{2x} = \epsilon,
\]

It follows that \(\{x_k\}\) does not converge with respect to \(d_1\) to any \(x \in \mathbb{N}\).

• \((\mathbb{N}, d_2)\) is complete. Any Cauchy sequence \(\{x_k\}\) is constant from some point on, since \(d_2(x_j, x_k) < 1/2\) implies that \(x_j = x_k\). Hence, every Cauchy sequence converges.
8. Define a two-form in $\mathbb{R}^3$ by
\[ \omega = (x^2 + y^2 + z^2) \, dx \wedge dy. \]

Let
\[ I = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \pi/2, \quad 0 \leq v \leq 2\pi\} \]
and define the two-cell $\phi : I \rightarrow \mathbb{R}^3$ (a half-sphere) by $\phi(u, v) = (x, y, z)$ where
\[ x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u. \]

Evaluate
\[ \int_{\phi} \omega. \]

Solution.

- From the definition of the integral of a differential form, we have
\[ \int_{\phi} \omega = \int_{I} (x^2 + y^2 + z^2) \, \frac{\partial(x, y)}{\partial(u, v)} \, dudv. \]

- On the surface, we have
\[ x^2 + y^2 + z^2 = \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u = 1. \]

- The Jacobian is
\[ \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \cos u \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u \cos v \end{vmatrix} = \cos u \sin u. \]

- It follows that
\[ \int_{\phi} \omega = \int_{I} \cos u \sin u \, dudv. \]
Evaluating this integral by use of Fubini’s theorem, we get

\[
\int_{\phi} \omega = \int_{0}^{\pi/2} \left( \int_{0}^{2\pi} \cos u \sin u \, dv \right) \, du \\
= 2\pi \int_{0}^{\pi/2} \cos u \sin u \, du \\
= 2\pi \left[ \frac{1}{2} \sin^2 u \right]_{0}^{\pi/2} \\
= \pi.
\]