1. Suppose that $X$ is a linear space of dimension $n$, and $E = \{e_1, \ldots, e_n\}$, $F = \{f_1, \ldots, f_n\}$ are two bases of $X$. Prove that there is a unique invertible $n \times n$ matrix $[s_{ij}]$ such that if a vector $x \in X$ has components $[a_i]$ with respect to $E$ and components $[b_j]$ with respect to $F$, meaning that

$$x = \sum_{i=1}^{n} a_i e_i, \quad x = \sum_{j=1}^{n} b_j f_j,$$

then

$$a_i = \sum_{j=1}^{n} s_{ij} b_j.$$

2. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with components $f = (f_1, f_2, \ldots, f_m)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$. Prove that $f(x) \to L$ as $x \to a$, where $a \in \mathbb{R}^n$ and $L = (L_1, L_2, \ldots, L_m) \in \mathbb{R}^m$, if and only if $f_i(x) \to L_i$ as $x \to a$ for every $1 \leq i \leq m$.

3. Suppose that $A, B \in L(X, Y)$. Prove that

$$\|A + B\| \leq \|A\| + \|B\|.$$

4. Prove that the following two definitions of a closed set $F \subset \mathbb{R}^n$ are equivalent: (a) $F^c$ is open; (b) the limit of every convergent sequence in $F$ belongs to $F$.

5. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f((x, y)) = (x + y, xy, x^3 + y^2).$$

Prove that $f$ is differentiable at $(1, 1)$ and compute $f''((1, 1))$. 
