Homework Problems: Set 6  
Math 127C, Spring 2006

1. Let \( Q = \mathbb{Q} \cap [0,1] \) denote the rational numbers in the interval \([0,1]\), and \( I = [0,1] \times [0,1] \) the unit square in \( \mathbb{R}^2 \). For \( x \in Q \), we write \( x = p/q \) in lowest terms (meaning that \( 0 \leq p \leq q \) are relatively prime integers), and define the set \( S(x) \subset I \) by

\[
S(x) = \{ (m/q, n/q) : m = 0, 1, 2, \ldots , p, n = 0, 1, 2, \ldots , p \}.
\]

We define \( S \subset I \) by

\[
S = \bigcup_{x \in Q} S(x),
\]

and \( f : I \to \mathbb{R} \) by

\[
f(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in S \\
1 & \text{if } (x, y) \notin S
\end{cases}
\]

Show that both iterated integrals

\[
\int_0^1 \left( \int_0^1 f(x, y) \, dy \right) \, dx, \int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy
\]

exist and are equal to 1, but \( f \) is not Riemann integrable on \( I \).

2. The convolution of two functions \( f, g : \mathbb{R} \to \mathbb{R} \) is the function \( f * g : \mathbb{R} \to \mathbb{R} \) defined by

\[
f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy,
\]

provided that the integral on the right hand side exists for every \( x \in \mathbb{R} \).

(a) If \( f, g \) are continuous functions with compact support, prove that \( f * g \) is also a continuous function with compact support.

(b) We define the \( L^1 \)-norm of a function \( f : \mathbb{R} \to \mathbb{R} \) by

\[
\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx.
\]

If \( f, g \) are continuous functions with compact support, prove that

\[
\|f * g\|_1 \leq \|f\|_1 \|g\|_1.
\]
3. If \( f : [0, \infty) \to \mathbb{R} \) is a continuous function, we define the improper Riemann integral of \( f \) on \([0, \infty)\) by

\[
\int_0^\infty f(x) \, dx = \lim_{a \to \infty} \int_0^a f(x) \, dx,
\]

assuming that this limit exists.

(a) By the use of two integrations by parts, or otherwise, derive the indefinite integral

\[
\int \sin x e^{-rx} \, dx = -\left( \frac{\cos x + r \sin x}{1 + r^2} \right) e^{-rx} + C,
\]

where \( r \) is a constant.

(b) Evaluate the integral of the function

\[ f(x, y) = \sin x e^{-xy} \]

over the rectangle \( I = [0, a] \times [0, b] \) in two different ways, and let \( b \to \infty \).

Deduce that

\[
\int_0^a \sin x \frac{1}{x} \, dx = \frac{\pi}{2} - \int_0^\infty (\cos a + y \sin a) \frac{e^{-ay}}{1 + y^2} \, dy.
\]

(c) Prove that

\[
\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]