1. For \( x, y \in \mathbb{R}^n \) with \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \), define
\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.
\]
(a) Prove that \( d \) is a metric on \( \mathbb{R}^n \).
(b) Prove that \( \mathbb{R}^n \) is complete with respect to the metric \( d \). (You can assume that \( \mathbb{R} \) is complete with respect to the standard metric.)

Solution.

• (a) It is immediate that \( d(x, y) = d(y, x) \) and \( d(x, y) \geq 0 \). Moreover, \( d(x, y) = 0 \) if and only if \( x_j = y_j \) for every \( 1 \leq j \leq n \), meaning that \( x = y \).

• To prove the triangle inequality, note that if \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n \), then
\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|
= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|
+ \cdots + |x_n - z_n + z_n - y_n|
\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|
+ \cdots + |x_n - z_n| + |z_n - y_n|
\leq d(x, z) + d(z, y).
\]
This verifies that \( d \) is a metric.

• (b) Suppose that \( \{x^{(k)} : k = 1, 2, 3, \ldots\} \) is a Cauchy sequence in \( \mathbb{R}^n \) with respect to \( d \). We write \( x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \).
For each \(1 \leq j \leq n\), we have
\[
|x_j - y_j| \leq d(x, y)
\]
so it follows that \(\{x_j^{(k)} : k = 1, 2, 3, \ldots\}\) is a Cauchy sequence in \(\mathbb{R}\) for each \(j\). Since \(\mathbb{R}\) is complete there exists \(x_j \in \mathbb{R}\) such that \(x_j^{(k)} \to x_j\) as \(k \to \infty\).

- Let \(x = (x_1, x_2, \ldots, x_n)\). Then we claim that \(x^{(k)} \to x\) as \(k \to \infty\) with respect to \(d\), which proves that \(\mathbb{R}^n\) is complete with respect to \(d\).

- Let \(\epsilon > 0\) be given. Since \(x_j^{(k)} \to x_j\), there exists \(K_j\) such that
\[
\left| x_j^{(k)} - x_j \right| < \frac{\epsilon}{n} \quad \text{for all } k > K_j.
\]
Let \(K = \max\{K_1, K_2, \ldots, K_n\}\). Then if \(k > K\), we have
\[
d(x^{(k)}, x) = \left| x_1^{(k)} - x_1 \right| + \left| x_2^{(k)} - x_2 \right| + \ldots + \left| x_n^{(k)} - x_n \right|
\leq \frac{\epsilon}{n} + \frac{\epsilon}{n} + \ldots + \frac{\epsilon}{n}
\leq \epsilon,
\]
which shows that \(x^{(k)} \to x\) as \(k \to \infty\).
2. (a) Let $I = [0, 1] \times [0, 1]$ be the unit square in $\mathbb{R}^2$. State Fubini’s theorem for the Riemann integral of a function $f : I \to \mathbb{R}$.
(b) Define the function $g : I \to \mathbb{R}$ by $g(x, y) = 0$ if $y \neq 1/2$, and

$$g(x, 1/2) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that $g$ is Riemann integrable on $I$.
(c) For $y \in [0, 1]$, define $g^y : [0, 1] \to \mathbb{R}$ by $g^y(x) = g(x, y)$. Find the upper and lower Riemann integrals of $g^y$,

$$\int_0^1 g^y(x) \, dx, \quad \int_0^1 g^y(x) \, dx.$$

Is $g^y$ is Riemann integrable on $[0, 1]$ for every $y \in [0, 1]$?
(d) Does $g$ satisfy Fubini’s theorem for the Riemann integral?

**Solution.**

- Fubini’s theorem states that if $f : I \to \mathbb{R}$ is Riemann integrable on $I$, then the upper and lower Riemann integrals with respect to $x$

$$\int_0^1 f(x, y) \, dx, \quad \int_0^1 f(x, y) \, dx.$$

are Riemann integrable with respect to $y$, and

$$\int_I f \, dxdy = \int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy = \int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy.$$

(Of course, the same statement is true with the roles of $x$ and $y$ exchanged.)

- (b) Given any $\epsilon > 0$, consider the partition $\mathcal{P}_\epsilon$ of $I$ into three rectangles,

$$R_1 = \{(x, y) : 0 \leq x \leq 1, \quad 0 \leq y \leq 1/2 - \epsilon\},$$

$$R_2 = \{(x, y) : 0 \leq x \leq 1, \quad 1/2 - \epsilon \leq y \leq 1/2 + \epsilon\},$$

$$R_3 = \{(x, y) : 0 \leq x \leq 1, \quad 1/2 + \epsilon \leq y \leq 1\}.$$
• The infimum and supremum of \( g \) on \( R_1, R_3 \) are 0, while the infimum of \( g \) on \( R_2 \) is 0 and the supremum of \( g \) is 1. The area of \( R_2 \) is \( 2\epsilon \). It follows that the lower and upper Riemann sums of \( g \) with respect to \( \mathcal{P}_\epsilon \) are \( \mathcal{L}(g, \mathcal{P}_\epsilon) = 0 \) and \( \mathcal{U}(g, \mathcal{P}_\epsilon) = 2\epsilon \), respectively.

• Since
\[
\mathcal{L}(g, \mathcal{P}_\epsilon) \leq \int_I g \, dx \, dy \leq \inf_{\epsilon > 0} \mathcal{U}(g, \mathcal{P}_\epsilon)
\]
and
\[
\mathcal{L}(g, \mathcal{P}_\epsilon) = 0, \quad \inf_{\epsilon > 0} \mathcal{U}(g, \mathcal{P}_\epsilon) = 0,
\]
we conclude that
\[
\int_I g \, dx \, dy = 0.
\]

• Since
\[
\mathcal{L}(g, \mathcal{P}_\epsilon) \leq \int_I g \, dx \, dy \leq \int_I g \, dx \, dy,
\]
we see that
\[
\int_I g \, dx \, dy = 0.
\]
Hence, the upper and lower Riemann integrals are the same, and \( g \) is Riemann integrable on \( I \) with
\[
\int_I g \, dx \, dy = 0.
\]

• (c) The function \( g^y \) is not Riemann integrable on \([0, 1]\) when \( y = 1/2 \), since
\[
g^{1/2}(x) = \begin{cases} 
 1 & \text{if } x \in \mathbb{Q} \\
 0 & \text{if } x \notin \mathbb{Q}
\end{cases},
\]
and
\[
\int_0^1 g^{1/2} \, dx = 0, \quad \int_0^1 g^{1/2} \, dx = 1.
\]

• For \( y \neq 1/2 \), we have \( g^y = 0 \), which is Riemann integrable, with
\[
\int_0^1 g^y \, dx = \int_0^1 g^y \, dx = 0.
\]
• Fubini’s theorem holds for \( g \) (as it must, since \( g \) is Riemann integrable on \( I \)).

• We can verify this explicitly. The lower \( x \)-integral is

\[
\int_0^1 g^y \, dx = 0 \quad \text{for all } 0 \leq y \leq 1,
\]

which is Riemann integrable on \([0, 1]\) with

\[
\int_0^1 \left( \int_0^1 g^y \, dx \right) \, dy = 0.
\]

The upper \( x \)-integral is

\[
\int_0^1 g^y \, dx = \begin{cases} 
1 & \text{if } y = 1/2 \\
0 & \text{if } y \neq 1/2
\end{cases}.
\]

Although discontinuous, this function is Riemann integrable on \([0, 1]\) (use a partition that includes an arbitrarily small interval containing 1/2), and

\[
\int_0^1 \left( \int_0^1 g^y \, dx \right) \, dy = 0.
\]
3. For $x, u, v \in \mathbb{R}$, consider the equations

\[
\begin{align*}
  u + v^3 - x &= 0, \\
  u^2 + 3xv - x^3 + x - 1 &= 0.
\end{align*}
\]

(a) Verify that: (i) if $x = 1$, then a solution for $(u, v)$ is $(u, v) = (1, 0)$; (ii) if $x = 2$, then a solution for $(u, v)$ is $(u, v) = (1, 1)$.

(b) For each of the solutions (i) and (ii) in (a), say whether or not the implicit function theorem guarantees that there is a unique local solution of the equations for $u = f(x), v = g(x)$. If the implicit function theorem does apply, give a complete and precise statement of what it implies.

Solution.

• (a) Easy to check.

• (b) The equations are of the form $F(u, x) = 0$, where $u = (u, v)$ and $F: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by $F(u, x) = (F(u, v, x), G(u, v, x))$ with

\[
F(u, v, x) = u + v^3 - x, \quad G(u, v, x) = u^2 + 3xv - x^3 + x - 1.
\]

• The partial derivatives of these functions exist and are continuous in $\mathbb{R}^2 \times \mathbb{R}$, so $F$ is continuously differentiable. The implicit function theorem therefore applies at points where $D_u F$ is invertible.

• We compute that the matrix of the derivative with respect to $u$ is

\[
[D_u F(u, v, x)] = \begin{bmatrix}
1 & 3v^2 \\
2u & 3x
\end{bmatrix}.
\]

• For (i), we get

\[
[D_u F(1, 0, 1)] = \begin{bmatrix}
1 & 0 \\
2 & 3
\end{bmatrix}.
\]

This matrix has non-zero determinant, so it is invertible.

• The implicit function theorem implies that there are open sets $V \subset \mathbb{R}$ containing 1 and $U \subset \mathbb{R}^2$ containing $(1, 0)$ such that for every $x$ in $V$, the equation $F(u, x) = 0$ has a unique solution for $u$ that belongs to $U$. Moreover, the one-to-one onto function $f: V \to U$ whose value at $x \in V$ is the unique solution $u \in U$ is continuously differentiable.
For (ii), we get

\[
[DuF(1, 1, 2)] = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.
\]

This matrix has zero determinant, so the derivative is not invertible, and the implicit function theorem does not lead to any conclusion.
4. Define \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by \( \phi(u, v) = (x, y) \), where
\[
\begin{align*}
x &= e^u \cos v, \\
y &= e^u \sin v.
\end{align*}
\]

(a) Why is \( \phi \) continuously differentiable? Compute the Jacobian
\[
J = | \det \phi' |
\]
of \( \phi \), and show that \( \phi'(u, v) \) is nonsingular for every \( (u, v) \in \mathbb{R}^2 \).

(b) Define open sets \( E, U \) in \( \mathbb{R}^2 \) by
\[
E = \{(u, v) \in \mathbb{R}^2 : 0 < u < 1, \ 0 < v < \pi/2 \},
\]
\[
U = \{(x, y) \in \mathbb{R}^2 : x > 0, \ y > 0, \ 1 < x^2 + y^2 < e^2 \}.
\]
Show that \( \phi : E \rightarrow U \) is one-to-one and onto.

(c) Suppose that \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a continuous function with compact support contained in \( U \). Use the change of variables theorem to express
\[
\int_{\mathbb{R}^2} f(x, y) \, dxdy
\]
as an integral with respect to \( (u, v) \).

Solution.

- (a) \( \phi \) is continuously differentiable since the partial derivatives of its component functions exist and are continuous everywhere.

- The Jacobian is
\[
J = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}.
\]
Since this is nonzero, the derivative is invertible.

- (b) \( \phi \) maps \((u, v)\) to the point with polar coordinates \((r, \theta)\) where \( r = e^u \) and \( \theta = v \). Thus \( \phi(E) \) consists of the points with \( 1 < r < e \) and \( 0 < \theta < \pi/2 \), which is \( U \). If \( \phi(u, v) = \phi(u', v') \), then elimination of \( v \) and \( v' \) implies that \( e^{2u} = e^{2u'} \), so \( u = u' \); and then \( \cos v = \cos v' \) implies that \( v = v' \) since \( 0 < v, v' < \pi/2 \). Thus, \( \phi : E \rightarrow U \) is one-to-one and onto.
(c) Define \( g : \mathbb{R}^2 \to \mathbb{R} \) by \( g(u, v) = (f \circ \phi)(u, v) \) if \((u, v) \in E\) and \(g(u, v) = 0\) otherwise. Explicitly,

\[
g(u, v) = \begin{cases} f(e^u \cos v, e^u \sin v) & \text{if } (u, v) \in E \\ 0 & \text{if } (u, v) \notin E \end{cases}
\]

Then \( g \) is a continuous function with compact support contained in \( E \). By the change of variables theorem,

\[
\int_{\mathbb{R}^2} f(x, y) \, dxdy = \int_{\mathbb{R}^2} g(u, v)e^{2u} \, dudv.
\]

Alternatively, defining the closed rectangle \( J = \overline{E} \) by

\[
J = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq 1, \quad 0 \leq v \leq \pi/2\},
\]

and letting \( I \) be any closed rectangle such that \( U \subset I \), we have

\[
\int_{I} f(x, y) \, dxdy = \int_{J} f \left( e^u \cos v, e^u \sin v \right) e^{2u} \, dudv.
\]

There is one slightly tricky detail: it is not true that

\[
\int_{\mathbb{R}^2} f(x, y) \, dxdy = \int_{\mathbb{R}^2} f \left( e^u \cos v, e^u \sin v \right) e^{2u} \, dudv.
\]

The integral on the right-hand side of this equation is not even well-defined since the integrand does not have compact support (except in the trivial case when \( f = 0 \)).

The difficulty here comes from the fact that although \( \phi \) is locally one-to-one, it is not globally one-to-one. For any \( n \in \mathbb{Z} \), \( \phi \) maps the set

\[
E_n = \{(u, v) \in \mathbb{R}^2 : 0 < u < 1, \quad 2\pi n < v < 2\pi n + \pi/2\}
\]

one-to-one and onto \( U \). Thus, the fact that \( f \) has compact support contained in \( U \) does not imply that \( f \circ \phi \) has compact support contained in \( E \). Instead, the support of \( f \circ \phi \) is contained in the unbounded (non-compact) set

\[
\bigcup_{n=-\infty}^{\infty} E_n.
\]