1. Show that

\[ d(x, y) = \frac{|x - y|}{1 + |x - y|} \]

is a metric on \( \mathbb{R} \). Is \( \mathbb{R} \) complete with respect to this metric?

**Solution.**

- The properties that \( d(x, y) = d(y, x) \), and \( d(x, y) \geq 0 \), with \( d(x, y) = 0 \) if and only if \( x = y \), are obvious. The only nontrivial part in the proof that \( d \) is a metric is the triangle inequality.

- Define \( f : [0, \infty) \to \mathbb{R} \) by

\[ f(t) = \frac{t}{1 + t}. \]

Then

\[ d(x, y) = f(|x - y|). \]

- The function \( f \) is monotonic increasing, since it is differentiable and

\[ f'(t) = \frac{1}{(1 + t)^2} > 0. \]

Hence, if \( 0 \leq u \leq v \), then \( f(u) \leq f(v) \).

- Suppose that \( s, t \geq 0 \). Then

\[
\begin{align*}
  f(s + t) &= \frac{s + t}{1 + s + t} \\
  &= \frac{s}{1 + s + t} + \frac{t}{1 + s + t} \\
  &\leq \frac{s}{1 + s} + \frac{t}{1 + t} \\
  &\leq f(s) + f(t).
\end{align*}
\]
• Suppose that $x, y, z \in \mathbb{R}$. Let

\[ u = |x - y|, \quad v = |x - z| + |z - y|, \quad s = |x - z|, \quad t = |z - y|. \]

Then $u \leq v$, by the triangle inequality on $\mathbb{R}$ with the Euclidean metric, and $v = s + t$. It then follows from the definition of $d$ and the properties of $f$ proved above that

\[
\begin{align*}
    d(x, y) &= f(u) \\
    &\leq f(v) \\
    &\leq f(s + t) \\
    &\leq f(s) + f(t) \\
    &\leq d(x, z) + d(z, y).
\end{align*}
\]

Thus, $d$ satisfies the triangle inequality, so it defines a metric on $\mathbb{R}$.

• Note that, with respect to the metric $d$, the distance between any two points in $\mathbb{R}$ is less than 1.

• The metric space $(\mathbb{R}, d)$ is complete. To prove this, we establish upper and lower bounds of $d(x, y)$ in terms of the Euclidean metric $e(x, y) = |x - y|$, and then deduce the completeness of $(\mathbb{R}, d)$ from the completeness of $(\mathbb{R}, e)$.

• In one direction, we have

\[
    d(x, y) = \frac{|x - y|}{1 + |x - y|} \leq |x - y|.
\]

• In the other direction, note that since $f$ is monotone increasing and $f(1/2) = 2/3$, we have

\[
    d(x, y) \leq 2/3 \text{ if and only if } |x - y| \leq 1/2.
\]

Thus, if $d(x, y) \leq 2/3$ then

\[
\begin{align*}
    d(x, y) &= \frac{|x - y|}{1 + |x - y|} \\
    &\geq \frac{|x - y|}{1 + 1/2} \\
    &\geq \frac{2}{3} |x - y|.
\end{align*}
\]
Now suppose that \( \{x_n\} \) is a Cauchy sequence in \( \mathbb{R} \) with respect to the metric \( d \). Given \( \epsilon > 0 \), let

\[
\epsilon' = \min \left( \frac{2}{3}, \frac{2}{3} \epsilon \right) > 0.
\]

Since \( \{x_n\} \) is Cauchy, we can choose \( N \) such that

\[
d(x_n, x_m) < \epsilon' \quad \text{for all } n, m > N.
\]

Then, for all \( n, m > N \), we have \( d(x_n, x_m) < 2/3 \), and it follows that

\[
|x_n - x_m| \leq \frac{3}{2} d(x_n, x_m) < \frac{3}{2} \epsilon' \leq \epsilon.
\]

This proves that \( \{x_n\} \) is a Cauchy sequence in \((\mathbb{R}, e)\). Since \((\mathbb{R}, e)\) is complete, there exists \( x \in \mathbb{R} \) such that \( |x_n - x| \to 0 \) as \( n \to \infty \).

Since \( d(x_n, x) \leq |x_n - x| \), we see that \( d(x_n, x) \to 0 \) as \( n \to \infty \). Thus, there exists \( x \in \mathbb{R} \) such that \( x_n \to x \) as \( n \to \infty \) with respect to \( d \), so \((\mathbb{R}, d)\) is complete.

2. Does the equation

\[
x^5 + y^5 + xy + 4 = 0
\]

define an implicit function \( x = g(y) \) locally near the point \((x, y) = (-2, 2)\)? Explain your answer.

Solution.

- The equation is of the form \( f(x, y) = 0 \) where \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is given by

\[
f(x, y) = x^5 + y^5 + xy + 4.
\]

This function is continuously differentiable in \( \mathbb{R} \times \mathbb{R} \) since its partial derivatives exist and are continuous everywhere.

- The partial derivative \( D_x f(x, y) = 5x^4 + y \) is nonsingular at \((x, y) = (-2, 2)\), so the implicit function theorem implies that the equation defines an implicit function \( g : J \to I \), where \( J, I \) are open sets containing \(-2, -2\) respectively, and for each \( y \in J \), \( x = g(y) \) is the unique solution of \( f(x, y) = 0 \) that lies in \( I \).
3. Suppose that $1/2 \leq a \leq 3/2$. Define a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(x) = x + \frac{1}{2} (a - x^2)$$

Find a closed bounded interval $I \subset \mathbb{R}$ containing 1 such that $\phi : I \rightarrow I$ is a contraction. If $x_0 \in I$, what do the iterates

$$x_{n+1} = x_n + \frac{1}{2} (a - x_n^2)$$

converge to as $n \rightarrow \infty$?

Solution.

- Check that if $I = [1/2, 3/2]$, then $\phi : I \rightarrow I$ maps $I$ into itself.
- Check that $|\phi'(x)| \leq 1/2$ for $x \in I$, so that $\phi$ is a contraction on $I$.
- Conclude that $\phi$ has a unique fixed point $\bar{x}$ in $I$, which must equal $\sqrt{a}$. Hence the iterates $x_n$ converge to $\sqrt{a}$ as $n \rightarrow \infty$.

4. Use the change of variables formula to transform

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} \, dx \, dy$$

into an integral with respect to polar coordinates $(r, \theta)$, where

$$x = r \cos \theta, \quad y = r \sin \theta.$$ 

Deduce the value of

$$\int_0^\infty e^{-x^2} \, dx$$

Justify your steps.

Solution.

- We will just give the formal computation. Note that, given the theorems shown in class, justification is required for the use of integrals over $[0, \infty)$, and in the change of variables formula, because the transformation between polar and cartesian coordinates is not one-to-one at the origin, and the Jacobian vanishes there. To resolve these problems, consider integrals over an annulus $\epsilon^2 \leq x^2 + y^2 \leq a^2$, then let $a \rightarrow \infty$ and $\epsilon \rightarrow 0$. 

• Fubini’s theorem implies that
\[
\int_0^\infty \int_0^\infty e^{-x^2-y^2} \, dx \, dy = \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-y^2} \, dy \right) = \left( \int_0^\infty e^{-x^2} \, dx \right)^2.
\]

• The Jacobian of the transformation from polar to Cartesian coordinates is
\[
J = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta 
\end{vmatrix} = r.
\]

• By the change of variables theorem, followed by Fubini’s theorem,
\[
\int_0^\infty \int_0^\infty e^{-x^2-y^2} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta
= \frac{\pi}{2} \left[ -\frac{1}{2} e^{-r^2} \right]_0^\infty
= \frac{\pi}{4}.
\]

• Since these integrals are equal, it follows that
\[
\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}.
\]