1. If \( S = \{x_1, \ldots, x_n\} \) and \( x, y \in \text{span} \, S \), then

\[
x = \sum_{j=1}^{n} a_j x_j, \quad y = \sum_{j=1}^{n} b_j x_j
\]

for some \( a_j, b_j \in \mathbb{R} \). Hence

\[
x + y = \sum_{j=1}^{n} (a_j + b_j) x_j \in \text{span} \, S,
\]

and for \( c \in \mathbb{R} \)

\[
cx = \sum_{j=1}^{n} (ca_j) x_j \in \text{span} \, S,
\]

so \( \text{span} \, S \) is a vector space.

2. Using the definition of \( BA \) as the composition and the linearity of \( A, B \) we have for \( x, y \in X \) and \( c \in \mathbb{R} \) that

\[
BA(x + y) = B(A(x + y)) = B(Ax + Ay) = B(Ax) + B(Ay) = BAx + BAy,
\]

\[
BA(cx) = B(A(cx)) = B(cAx) = cB(Ax) = cBAx,
\]

which proves that \( BA \) is linear.

Suppose that \( A : X \to Y \) is an invertible linear map. Then \( A^{-1} y = x \) if and only if \( Ax = y \). Suppose that \( c \in \mathbb{R} \). Then \( A^{-1}(cy) = z \) implies that
\[Az = cy.\] If \(c = 0\), then \(z = 0\) since \(A\) is one-to-one, so \(A^{-1}(0y) = 0y\). Otherwise,

\[
A\left(\frac{1}{c}z\right) = y,
\]

since \(A\) is linear, so

\[
A^{-1}y = \frac{1}{c}z = \frac{1}{c}A^{-1}(cy).
\]

Hence,

\[
A^{-1}(cy) = cA^{-1}y.
\]

If \(A^{-1}y_1 = x_1, A^{-1}y_2 = x_2\), then \(Ax_1 = y_1, Ax_2 = y_2\). Since \(A\) is linear,

\[
A(x_1 + x_2) = Ax_1 + Ax_2 = y_1 + y_2,
\]

so

\[
A^{-1}(y_1 + y_2) = x_1 + x_2 = A^{-1}y_1 + A^{-1}y_2.
\]

It follows that \(A^{-1}\) is a linear map.

If \(A : X \to Y\) is invertible, then \(A^{-1} : Y \to X\) is one-to-one since \(A\) is one-to-one, and \(A^{-1}\) is onto since its range is the domain \(X\) of \(A\). Hence, \(A^{-1}\) is invertible (with inverse equal to \(A\)).

3. Suppose \(Ax_1 = Ax_2\). Then the linearity of \(A\) implies that \(A(x_1 - x_2) = 0\), so \(x_1 - x_2 = 0\), and \(x_1 = x_2\). That is, \(A\) is one-to-one.

4. Suppose \(A : X \to Y\) is a linear map. If \(y \in R(A)\), there exists \(x \in X\) such that \(Ax = y\). If \(c \in \mathbb{R}\), then \(A(cx) = cy\), so \(cy \in R(A)\). Similarly, if \(y_1, y_2 \in R(A)\), there exists \(x_1, x_2 \in X\) such that \(Ax_1 = y_1, Ax_2 = y_2\). Since \(A\) is linear,

\[
A(x_1 + x_2) = y_1 + y_2,
\]

so \(y_1 + y_2 \in R(A)\). Hence the range \(R(A) \subset Y\) is a vector space.

If \(x \in N(A)\), then \(Ax = 0\). If \(c \in \mathbb{R}\), then

\[
A(cx) = cAx = 0,
\]
so \( c \mathbf{x} \in \mathcal{N}(A) \). If \( \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A) \), then

\[
A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0},
\]

so \( \mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A) \). Hence the nullspace \( \mathcal{N}(A) \subset X \) is a linear space.