6. The quotient rule (Theorem 5.3 (c) in Rudin) implies that the partial derivatives of \( f \) exist at \((x, y) \neq (0, 0)\), with

\[
D_1 f(x, y) = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2},
\]
\[
D_2 f(x, y) = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}.
\]

Since \( f(x, 0) = f(0, y) = 0 \) for all \( x, y \in \mathbb{R} \), the partial derivatives of \( f \) exist at \((0, 0)\), with

\[
D_1 f(0, 0) = D_2 f(0, 0) = 0.
\]

**Remark.** Note that although the partial derivatives of \( f \) exist for all \((x, y) \in \mathbb{R}^2\), they are not continuous at \((0, 0)\); for example, if \( y \neq 0 \) then \( D_1 f(0, y) = 1/y \), so \( D_1 f \) is unbounded in any neighborhood of the origin.

8. For \( h \in \mathbb{R}^n \), there is an open interval \( I \subset \mathbb{R} \) containing the origin such that \( x + th \in E \) for \( t \in I \). Define \( g : I \to \mathbb{R} \) by

\[
g(t) = f(x + th).
\]

Since \( f \) is differentiable at \( x \), \( g \) is differentiable at 0 and

\[
g'(0) = f'(x)h.
\]

Since \( f \) has a local maximum at \( x \), \( g \) has a local maximum at 0, and therefore \( g'(0) = 0 \) (by Theorem 5.8 in Rudin). It follows that \( f'(x)h = 0 \) for every \( h \in \mathbb{R}^n \), meaning that \( f'(x) = 0 \).

9. Choose \( a \in E \), and define

\[
U = \{x \in E : f(x) = f(a)\},
\]
\[
V = \{x \in E : f(x) \neq f(a)\}.
\]

Then \( E \) is the disjoint union of \( U \) and \( V \), and \( U \) is nonempty. We claim that \( U, V \) are open subsets of \( E \) (or \( \mathbb{R}^n \)). The connectedness of \( E \) then implies that \( V \) is empty, so \( f \) is constant on \( E \).
The set $V$ is open since $f$ is continuous (implied by its differentiability) and the inverse image of an open set by a continuous function is open.

To show that $U$ is open, suppose $x \in U$. Since $E$ is open in $\mathbb{R}^n$, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset E$, where

$$B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x| < \epsilon\}.$$

The open ball $B_\epsilon(x)$ is convex. Since $f'$ is zero on $E$, the Corollary to Theorem 9.19 of Rudin implies that $f$ is constant on $B_\epsilon(x)$, and hence equal to $f(a)$ (since $f(x) = f(a)$). It follows that $B_\epsilon(x) \subset U$, so $U$ is open.

**Remark.** If the domain $E$ is convex, we can prove this result directly as in the proof of Theorem 9.19 of Rudin. For general open sets $E$, we have to do a little bit of point set topology. We have used the standard definition that a topological space is connected if it is not the disjoint union of two nonempty open sets. Rudin’s Definition 2.45 seems to be equivalent to this one, but weird.