Complex Analysis
Math 185A, Winter 2010
Final: Solutions

1. [25 pts] The Jacobian of two real-valued functions $u(x,y)$, $v(x,y)$ of (x,y) is defined by the determinant

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function of $z = x + iy$, prove that

$$J(x,y) = |f'(z)|^2 .$$

Solution.

- Since f is analytic,

  $$f' = u_x + iv_x = v_y - iu_y$$

  and u, v satisfy the Cauchy-Riemann equations

  $$u_x = v_y, \qquad v_x = -u_y.$$

- It follows that

  $$\begin{aligned} J &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= u_x v_y - u_y v_x \\ &= u_x^2 + v_x^2 \\ &= |f'|^2 . \end{aligned}$$

Remark. Note that the Jacobian is strictly positive when $f' \neq 0$, corresponding to the fact that a conformal map f is locally one-to-one. Also, J is always nonnegative which corresponds to the fact that analytic functions preserve orientations.

2. [25 pts] Define

$$f(z) = z + \frac{1}{z}.$$

(a) Find the image of the unit circle $|z| = 1$ under f.
(b) On what open sets $\Omega \subset \mathbb{C}$ is $f : \Omega \to \mathbb{C}$ a conformal map?

Solution.

- (a) If $|z| = 1$ then $z = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta.$$

  So f maps the unit circle to the interval $[-2, 2]$ on the real axis.

- (b) The function f is analytic except at $z = 0$. Its derivative is then

$$f'(z) = 1 - \frac{1}{z^2}$$

  which is nonzero unless $z = \pm 1$. Therefore f is conformal on any open set $\Omega \subset \mathbb{C}$ that does not contain any of the points $\{-1, 0, 1\}$.

Remark. Note that f has a simple zero at $z = \pm 1$, so it doubles the angle between curves at those points, where it 'flattens' the circle $|z| = 1$ to the interval $[-2, 2]$. This mapping is used in fluid mechanics to obtain the potential flow of an ideal fluid past a cylinder of radius one, which f maps to uniform flow past a flat plate of length four.

3. [25 pts] Let $\gamma : [0, \pi] \to \mathbb{C}$ with $\gamma(t) = 2e^{it}$ be the positively oriented semicircle in the upper half plane with center the origin and radius 2. Prove that

$$\left| \int_\gamma \frac{e^z}{z^2+1}\, dz \right| \le \frac{2\pi e^2}{3}.$$

(Do *not* try to evaluate the integral exactly.)

Solution.

- By the basic estimate for the modulus of a contour integral,

$$\begin{aligned} \left| \int_\gamma \frac{e^z}{z^2+1}\, dz \right| &\le \max_{z\in\gamma} \left| \frac{e^z}{z^2+1} \right| \cdot \text{length}(\gamma) \\ &\le \max_{z\in\gamma} \left| \frac{e^z}{z^2+1} \right| \cdot 2\pi. \end{aligned}$$

- If $z = x + iy$, then $|e^z| = e^x$. Since $x \le 2$ on γ, we have

$$|e^z| \le e^2 \qquad \text{for } z \in \gamma.$$

- By the reverse triangle inequality

$$\left|z^2+1\right| \ge |z|^2 - 1 \ge 3$$

  for $z \in \gamma$, so

$$\left| \frac{1}{z^2+1} \right| \le \frac{1}{3}.$$

- It follows that

$$\max_{z\in\gamma} \left| \frac{e^z}{z^2+1} \right| \le \frac{e^2}{3}$$

  and

$$\left| \int_\gamma \frac{e^z}{z^2+1}\, dz \right| \le \frac{2\pi e^2}{3}.$$

4. [25 pts] Suppose that $a, b, z \in \mathbb{C}$ are such that $az + b \neq 0$ and $|z| = 1$. Prove that

$$\left| \frac{\bar{b}z + \bar{a}}{az + b} \right| = 1.$$

Solution.

- Since $|z| = 1$, we have $z = 1/\bar{z}$ and

$$\bar{b}z + \bar{a} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{z}}.$$

- It follows that

$$\left| \frac{\bar{b}z + \bar{a}}{az + b} \right| = \frac{1}{|\bar{z}|} \frac{\left|\bar{a}\bar{z} + \bar{b}\right|}{|az + b|}.$$

- If $w = az + b$, then $\bar{w} = \bar{a}\bar{z} + \bar{b}$. Using the fact that $|\bar{w}| = |w|$ and $|\bar{z}| = 1$, we get

$$\left| \frac{\bar{b}z + \bar{a}}{az + b} \right| = 1.$$

5. [30 pts] Find the radius of convergence of the following power series:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^n}{n} z^n; \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{3n}; \qquad \text{(c)} \quad \sum_{n=0}^{\infty} n! z^{n!}.$$

Solution.

- (a) By the ratio test, the radius of convergence is given by

$$R = \lim_{n\to\infty} \frac{3^n/n}{3^{n+1}/(n+1)} = \lim_{n\to\infty} \frac{1+1/n}{3} = \frac{1}{3}.$$

  The series therefore converges in $|z| < 1/3$ and diverges in $|z| > 1/3$.

- (b) Let $w = z^3$. Then the series becomes

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} w^n$$

  By the ratio test, the radius of convergence in w is given by

$$R = \lim_{n\to\infty} \frac{2^n/n!}{2^{n+1}/(n+1)!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty.$$

  So the radius of convergence in z is also ∞. The series converges for every $z \in \mathbb{C}$.

- (c) Writing out the terms explicitly, we get

$$\sum_{n=0}^{\infty} n! z^{n!} = 1 + z + 2!z^2 + 3!z^6 + 4!z^{24} + \dots.$$

  If $|z| \le r < 1$, then the absolutely values of terms of this series are bounded by terms in series $\sum_{m=0}^{\infty} mr^m$, which is absolutely convergent by the ratio test. Therefore, the series converges in $|z| < 1$. If $|z| > 1$ then $n!z^{n!}$ does not approach zero as $n \to \infty$, so the series diverges. It follows that the radius of convergence is $R = 1$.

6. [30 pts] For $z \neq 0$, let

$$f(z) = z^5 e^{1/z^2}.$$

(a) Find the Laurent expansion of f at $z = 0$. (b) What is the residue of f at $z = 0$? (c) Where does the Laurent series converge? (d) What type of isolated singularity does f have at 0?

Solution.

- (a) Using the power series expansion for e^w, with $w = 1/z^2$, we get

$$\begin{aligned} z^5 e^{1/z^2} &= z^5 \left(1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots + \frac{1}{n!}\frac{1}{z^{2n}} + \ldots \right) \\ &= \cdots + \frac{1}{n!}\frac{1}{z^{2n-5}} + \ldots \frac{1}{3!}\frac{1}{z} + \frac{1}{2!} z + z^3 + z^5 \\ &= \sum_{n=-\infty}^{2} \frac{1}{(2-n)!} z^{2n+1}. \end{aligned}$$

- (b) The coefficient of $1/z$ is $1/3! = 1/6$, so

$$\operatorname{Res}(f; 0) = \frac{1}{6}.$$

- (c) The power series expansion of e^{1/z^2} converges for all $z \neq 0$, so the Laurent series converges for all $z \in \mathbb{C} \setminus \{0\}$.

- (d) The singularity is an essential singularity since there are infinitely many terms with negative exponents in the Laurent expansion.

7. [40 pts] From elementary calculus, we know that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \left[\tan^{-1} x\right]_{-R}^{R} = \pi.$$

Use contour integration and the method of residues to evaluate this integral and show that you get the same result.

Solution.

- Let $\gamma_R = \rho_R + \sigma_R$ be the positively oriented semicircle in the upper half plane with center the origin and radius R, where $\rho_R : [-R, R] \to \mathbb{C}$ is given by $\rho_R(t) = t$ and $\sigma_R : [0, \pi] \to \mathbb{C}$ is given by $\sigma_R(t) = Re^{it}$.

- The function
$$f(z) = \frac{1}{1+z^2}$$
is analytic except at $z = \pm i$, where it has simple poles (since $1+z^2$ has simple zeros). For $R > 1$, the pole at $z = i$ lies inside the simple closed curve γ_R, and the residue theorem implies that
$$\int_{\gamma_R} \frac{1}{1+z^2}\,dz = 2\pi i \operatorname{Res}(f; i).$$

- The residue of f at i is given by
$$\operatorname{Res}(f; i) = \lim_{z\to i} \frac{z-i}{1+z^2} = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}.$$
Thus,
$$\int_{\gamma_R} \frac{1}{1+z^2}\,dz = \pi. \tag{1}$$

- We have
$$\begin{aligned}\int_{\gamma_R} \frac{1}{1+z^2}\,dz &= \int_{\rho_R} \frac{1}{1+z^2}\,dz + \int_{\sigma_R} \frac{1}{1+z^2}\,dz \\ &= \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\sigma_R} \frac{1}{1+z^2}\,dz.\end{aligned} \tag{2}$$

- Using the basic estimate for contour integrals, and the reverse triangle inequality, we have for $R > 1$ that

$$\begin{aligned}\int_{\sigma_R} \frac{1}{1+z^2}\, dz &\leq \max_{z \in \sigma_R} \left| \frac{1}{1+z^2} \right| \cdot \operatorname{length}(\sigma_R) \\ &\leq \frac{\pi R}{R^2 - 1} \\ &\to 0 \qquad \text{as } R \to \infty.\end{aligned}$$

- Using (1) in (2) and taking the limit of the result as $R \to \infty$, we find that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$$

Extra Credit Problem. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is doubly-periodic with periods 1 and i, meaning that

$$f(z+1) = f(z), \quad f(z+i) = f(z) \qquad \text{for all } z \in \mathbb{C}.$$

Let $D = \{x + iy \in \mathbb{C} : 0 < x < 1,\, 0 < y < 1\}$ denote the open unit square and γ the boundary of D (see picture). It follows that $f(z+m+in) = f(z)$ for all integers $m, n \in \mathbb{Z}$, and f is determined on $\mathbb{C}$ by its values on D and γ.

(a) Prove that the only entire (*i.e.* analytic on $\mathbb{C}$) doubly-periodic functions are the constant functions.

(b) Prove that there are no doubly-periodic functions that have a single simple pole with nonzero residue in D and are analytic elsewhere in D and on γ. Hint. Consider the contour integral of f along γ.

(c) If f has two poles at $z_1, z_2 \in D$ and is analytic elsewhere in D and on γ, what can you say about the residues of f at z_1, z_2?

Solution.

- (a) If f is entire, then it is continuous on the compact set $\overline{D}$ and therefore bounded. By periodicity, f is bounded on $\mathbb{C}$ and therefore by Liouville's theorem, f is constant.

- (b) Let $\gamma = \gamma_b + \gamma_r + \gamma_t + \gamma_l$, where the four contours on the right-hand side are the bottom, right, top, and left sides of the unit square, respectively, with orientations corresponding to the positive (counter-clockwise) orientation of γ.

- Since f is periodic with period 1, its values on γ_b are equal to its values on γ_t; and since f is periodic with period i, its values on γ_l are equal to its values on on γ_r.

- The sides γ_b and γ_t have opposite orientations, as do γ_l and γ_r. It follows that

$$\int_{\gamma_t} f\,dz = -\int_{\gamma_b} f\,dz, \qquad \int_{\gamma_r} f\,dz = -\int_{\gamma_l} f\,dz.$$

Therefore

$$\int_{\gamma} f\,dz = \int_{\gamma_b} f\,dz + \int_{\gamma_r} f\,dz + \int_{\gamma_t} f\,dz + \int_{\gamma_l} f\,dz = 0.$$

- If f had a single simple pole at $z_0 \in D$, whose residue is necessarily nonzero (otherwise the singularity is removable), then the residue theorem would imply that

$$\int_\gamma f\,dz = 2\pi i \mathrm{Res}(f; z_0) \neq 0.$$

  This contradicts the fact that, as we have just shown, the contour integral must be zero by the double-periodicity of f.

- (c) If f has two poles at $z_1, z_2 \in D$, then by the residue theorem

$$\int_\gamma f\,dz = 2\pi i \left[\mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2)\right].$$

  Since this integral is zero, we must have

$$\mathrm{Res}(f; z_2) = -\mathrm{Res}(f; z_1).$$

Remark. More generally, one can consider doubly-periodic functions with periods 1 and $\tau \in \mathbb{C}$, where without loss of generality we can assume $\mathrm{Im}\,\tau > 0$. Meromorphic, doubly-periodic are called elliptic functions. By identifying the opposite sides of a period paralleogram

$$\overline{D} = \{s + t\tau : 0 \leq s \leq 1,\, 0 \leq t \leq 1\}$$

we may think of elliptic functions as meromorphic functions on a torus, or equivalently as holomorphic functions from the torus to the sphere. As indicated above, any such non-constant elliptic function must have at least two simple poles in a period, or one double pole. In that case, the elliptic function is a two-to-one mapping of the torus onto the sphere, which 'wraps' the torus twice around the sphere.

Perhaps the simplest elliptic function (from which all other elliptic functions can be constructed) is the Weierstrass $\wp$-function. This has a double pole at $z = 0$ which is extended by periodicity, in a way that gives a convergent series, to get

$$\wp(z; \tau) = \frac{1}{z^2} + \sum_{\substack{m, n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left[\frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2}\right].$$