Complex Variables

MAT 185A, Winter 2008

Midterm Exam

NAME..

I.D. NUMBER...

No books, notes, or calculators. Show all your work.

Question	Points	Score
1	30	
2	20	
3	20	
4	30	
Total	100	

NAME..

1. [30 pts.]

(1) Simplify $\left(\frac{1-i}{1+i}\right)^2$.

(2) Find all values of $log\,(4\,i)$.

(3) Prove the identity $1 - e^{i\theta} = -2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$ $(\theta \in \mathbb{R})$ and show the modulus and the argument lying in the interval $[0, 2\pi)$ of $1 - e^{i\theta}$, where $0 < \theta < \pi$.

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2. [20 pts.] For a given function $f(z) = 1 + 2iz - z^2$ and a line segment $\gamma(t) = 1 + it \ \ (0 \leq t \leq 1)$,

(1) solve $f(z) = 0$;

(2) evaluate the integral $\int\limits_{\gamma} f'(z)dz$.

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3. [20 pts.]

(1) Verify that the real and imaginary parts of $f(z) = z^3$ satisfy the Cauchy-Riemann equations.

(2) Find the maximum of $|e^{-2z}|$ on the disk $|z| \leq 1$.

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4. [30 pts.] Let the region $\mathcal{A} = \mathbb{C} \setminus \{\frac{1}{2}, -3\}$ and consider two closed curves:

$$\gamma_1 : \ \ \gamma_1(t) = \cos t + i \sin t \quad (0 \leq t \leq 2\pi),$$

$$\gamma_2 : \ \ \gamma_2(t) = 2\cos t + i \sin t \quad (0 \leq t \leq 2\pi).$$

(1) Which functions are analytic in $\mathcal{A}$ in the following functions ?

$$f(z) = (z+3)^2 e^{z^2}; \qquad g(z) = \frac{1}{\sin z}; \qquad h(z) = \frac{\cos(2\pi z)}{z - \frac{1}{2}}.$$

(2) Evaluate the integrals $\int\limits_{\gamma_1} f(z)\,\mathrm{d}z$ and $\int\limits_{\gamma_2} h(z)\,\mathrm{d}z$.

(3) Prove that the closed curves γ_1 and γ_2 are homotopic in the region $\mathcal{A}$.

Solutions

1(1) $\left(\frac{1-i}{1+i}\right)^2 = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$

1(2) $log\,(4\,i) = log\,|4i| + iarg\,(4i) + 2k\pi i = log\,4 + (\frac{\pi}{2} + 2k\pi)i, \;\; k \in \mathbb{Z}.$

1(3) $1 - e^{i\theta} = -e^{i\frac{\theta}{2}}(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}) = -2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$, the modulus is $|1-e^{i\theta}| = |-2i|\cdot|e^{i\frac{\theta}{2}}|\cdot|\sin\frac{\theta}{2}| = 2\sin\frac{\theta}{2}$, and the argument lying in the interval $[0, 2\pi)$ is $arg(1-e^{i\theta}) = arg(-2i) + arg(e^{i\frac{\theta}{2}}) + arg(\sin\frac{\theta}{2}) = \frac{3\pi}{2} + \frac{\theta}{2} + 0 = \frac{3\pi}{2} + \frac{\theta}{2}$, where $0 \le \frac{3\pi}{2} + \frac{\theta}{2} < 2\pi$.

2(1) Since $f(z) = 1 + 2iz - z^2 = (1+iz)^2$, the solution of the equation $f(z) = 0$ is $z = i$.

2(2) Since $\gamma(0) = 1$ and $\gamma(1) = 1 + i$, the integral

$$\int_\gamma f'(z)\,dz = f(\gamma(1)) - f(\gamma(0)) = f(1+i) - f(1).$$

Since $f(1+i) = -1$ and $f(1) = 2i$, we get $\int_\gamma f'(z)\,dz = -1 - 2i$.

3(1) Let $z = x + iy$. Then $z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. Denote $u = \mathrm{Re}(z^3)$ and $v = \mathrm{Im}(z^3)$. We have $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$. Thus, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x},$$

i.e., the Cauchy-Riemann equations hold.

3(2) The Maximum Modulus Theorem tells us that the maximum occurs on the boundary of the disk $|z| \le 1$. On the boundary $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$,

$$|e^{-2z}| = |e^{-2e^{i\theta}}| = |e^{-2\cos\theta - 2i\sin\theta}| = e^{-2\cos\theta}.$$

Thus, when $\theta = \pi$, i.e., $z = -1$, the maximum of $|e^{-2z}|$ occurs and the maximum of $|e^{-2z}|$ is e^2.

4(1) f is entire and h is analytic in $\mathbb{C}\setminus\{\frac{1}{2}\}$, thus f and h are analytic in $\mathcal{A}$. g is not differentiable at $z = 0$ and $0 \in \mathcal{A}$, thus g is not analytic in $\mathcal{A}$.

4(2) Since $f(z)$ is entire, by Cauchy's Theorem,

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = 0.$$

Since $\cos(2\pi z)$ is analytic in $\mathbb{C}$ and the winding number $I(\gamma_2, \frac{1}{2}) = 1$, by the Cauchy's Integral Formula,

$$\int_{\gamma_2} h(z)\,\mathrm{d}z = \int_{\gamma_2} \frac{\cos(2\pi z)}{z - \frac{1}{2}}\, dz = 2\pi i \cdot \cos\pi \cdot I(\gamma_2, \frac{1}{2}) = -2\pi i.$$

4(3) Take a continuous mapping $H: \ [0,1] \times [0, 2\pi] \mapsto \mathcal{A}$,

$$H(s,t) = (1+s)\cos t + i \sin t, \qquad 0 \le s \le 1, \quad 0 \le t \le 2\pi.$$

For each $s \in [0,1]$, $t \mapsto H(s,t)$ is a closed curve, and $H(0,t) = \cos t + i \sin t = \gamma_1(t)$ $(0 \le t \le 2\pi)$, $H(1,t) = 2\cos t + i \sin t = \gamma_2(t)$ $(0 \le t \le 2\pi)$. So γ_1 is homotopic to γ_2 in $\mathcal{A}$.