Complex Analysis
Math 185A, Winter 2010
Midterm: Solutions

1. If a, b are complex numbers such that $|a| < 1$, $|b| < 1$, prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1.$$

Solution.

- We have

$$\left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} = \frac{|a|^2 - (a\bar{b}+\bar{a}b) + |b|^2}{1-(a\bar{b}+\bar{a}b)+|a|^2|b|^2}. \tag{1}$$

- If $x, y < 1$, then

$$0 < (1-x)(1-y) = 1 - (x+y) + xy.$$

  It follows that

$$x + y < 1 + xy.$$

- Using this inequality with $x = |a|^2 < 1$ and $y = |b|^2 < 1$, we get

$$|a|^2 - (a\bar{b}+\bar{a}b) + |b|^2 < 1 - (a\bar{b}+\bar{a}b) + |a|^2|b|^2.$$

- Since $1 - (a\bar{b}+\bar{a}b) + |a|^2|b|^2 = |1-\bar{a}b|^2 > 0$, division of this inequality by the right hand side gives

$$\frac{|a|^2 - (a\bar{b}+\bar{a}b) + |b|^2}{1-(a\bar{b}+\bar{a}b)+|a|^2|b|^2} < 1.$$

  Using this in (1) proves the result.

2. Let $T \subset \mathbb{C}$ be the interior of the triangle with vertices at 0, 1, $1+i$ shown in the figure. Find the image of T under the map $w = z^2$ and draw a picture. Which angles of the triangle are preserved by the mapping?

Solution.

- The line segment from 0 to 1, $z = t$ where $0 \le t \le 1$, maps to the same line segment from 0 to 1, $w = s$ where $0 \le s \le 1$ and $s = t^2$.

- The line segment from 0 to $1 + i$, $z = te^{i\pi/4}$ where $0 \le t \le \sqrt{2}$, maps to the line segment from 0 to $2i$, $w = se^{i\pi/2} = is$ where $0 \le s \le 2$ and $s = t^2$.

- The line segment from 1 to $i + i$, $z = 1 + it$ with $0 \le t \le 1$ maps to $w = 1 - t^2 + 2it$. Writing $w = u + iv$ and eliminating t, we find that this is a segment of the parabola $4u = 4 - v^2$ from the vertex $w = 0$ to the intercept with the positive imaginary axis at $w = 2i$.

- The map takes the interior T of the triangle to the interior S of the region bounded by the line segments from 0 to 1 and 0 to $2i$ and the parabola from 1 to $2i$. For example, $e^{i\pi/6} \in T$ maps to $e^{i\pi/3} \in S$.

- The map is conformal except at the origin, since $z \mapsto z^2$ is analytic with nonzero derivative except at $z = 0$. The map therefor preserves the angles of the triangle at $z = 1$ and $z = 1 + i$, but doubles the angle at $z = 0$.

3. (a) State the Cauchy-Riemann equations satisfied by the real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$.
(b) Prove that there are two values of the constant $c \in \mathbb{R}$ such that

$$u(x, y) = e^{cy} \cos x$$

is the real part of an analytic function. Find the analytic function $f(z)$ in each case.

Solution.

- (a) The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x.$$

- (b) The real part of an analytic function is harmonic, so we must have

$$u_{xx} + v_{yy} = -e^{cy} \cos x + c^2 e^{cy} \cos x = \left(c^2 - 1\right) e^{cy} \cos x = 0.$$

  Hence, f is only analytic if $c = \pm 1$.

- If $c = 1$, the harmonic conjugate v of u satisfies

$$v_x = -e^y \cos x, \qquad v_y = -e^y \sin x.$$

  Integrating these equations, we get

$$v = -e^y \sin x + q(y), \qquad v = -e^y \sin x + p(x)$$

  where $p(x)$, $q(y)$ are real-valued functions of integration. It follows that $p(x) = q(y) = k$ where k is an arbitrary real constant, and therefore

$$f(z) = e^y \cos x - ie^y \sin x + ik = e^{-i(x+iy)} + ik.$$

  Hence,

$$f(z) = e^{-iz} + ik,$$

  where $k \in \mathbb{R}$ is an arbitrary constant of integration.

- Similarly, if $c = -1$, then the harmonic conjugate v of u satisfies

$$v_x = e^{-y} \cos x, \qquad v_y = -e^{-y} \sin x,$$

  and $v = e^{-y} \sin x + k$. It follows that $f(z) = e^{i(x+iy)} + ik$ or

$$f(z) = e^{iz} + ik.$$

- Alternatively, one can verify directly that these analytic functions have the correct real parts.

4. Let γ be the positively oriented circle with radius 1 and center i. Stating clearly any theorems you use, evaluate the following contour integrals:

$$\text{(a)} \int_\gamma \bar{z}\, dz; \qquad \text{(b)} \int_\gamma \frac{1}{z^2+2}\, dz; \qquad \text{(c)} \int_\gamma \frac{1}{z^2-2}\, dz.$$

Solution.

- (a) Since $\bar{z}$ is not an analytic function of z, we evaluate the contour integral directly. A parametrization of the curve is given by $\gamma : [0, 2\pi] \to \mathbb{C}$ where

$$\gamma(t) = i + e^{it}.$$

  It follows that

$$\begin{aligned}
\int_\gamma \bar{z}\, dz &= \int_0^{2\pi} \overline{\left(i + e^{it}\right)}\, ie^{it}\, dt \\
&= \int_0^{2\pi} \left(e^{it} + i\right)\, dt \\
&= \left[\frac{1}{i}e^{it} + it\right]_0^{2\pi} \\
&= 2\pi i.
\end{aligned}$$

- (b) We write the integral as

$$\int_\gamma \frac{1}{z^2+2}\, dz = \int_\gamma \frac{f(z)}{z - \sqrt{2}i}\, dz, \qquad f(z) = \frac{1}{z + \sqrt{2}i}.$$

  The function f is analytic inside and on γ, so Cauchy's integral formula implies that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - \sqrt{2}i}\, dz = f(\sqrt{2}i) = \frac{1}{2\sqrt{2}i}.$$

  Thus,

$$\int_\gamma \frac{1}{z^2+2}\, dz = \frac{\pi}{\sqrt{2}}.$$

- (c) The function $z \mapsto 1/(z^2 - 2)$ is analytic everywhere except at the points
$$z = \pm\sqrt{2},$$
which lie outside γ. Hence,
$$\int_\gamma \frac{1}{z^2 - 2}\, dz = 0$$
by Cauchy's theorem.

5. Define a function $f : A \to \mathbb{C}$ by

$$f(z) = e^{\sqrt{z}}, \qquad A = \{z \in \mathbb{C} : z \neq 0 \text{ and } \arg z \neq \pi\}$$

where we take the principle brach of the square root, and a function $g : B \to \mathbb{C}$ by

$$g(z) = \frac{e^z}{z}, \qquad B = \{z \in \mathbb{C} : z \neq 0\}.$$

Is there an analytic function $F : A \to \mathbb{C}$ such that $F' = f$ on A? Is there an analytic function $G : B \to \mathbb{C}$ such that $G' = g$ on B? Justify your answers, but do not try to find F or G explicitly if they exist.

Solution.

- The domain A is simply connected and the function f is analytic on A. Hence, by the 'antiderivative theorem' (*c.f.* Theorem 2.2.5 in the text) f has an antiderivative on A.

- To prove that A is simply connected, observe that the map $z \mapsto \sqrt{z}$, where we take the principle branch of the square root function, is a homeomorphism (*i.e.* a continuous, one-to-one, onto map with continuous inverse) of A onto the right-half plane $R = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The right-half plane is simply connected since it is convex, and any set that is homeomorphic to a simple connected set is also simply connected. (Note that A itself is not convex; for example, the line segment from $-1 + i \in A$ to $-1 - i \in A$ is not contained in A.)

- The domain B is connected but not simply connected, so we cannot apply the antiderivative theorem even though g is analytic on B. In fact, we claim that g does not have an antiderivative on B.

- To prove this, observe that if γ is the positively oriented unit circle centered at 0, which is a closed curve in B, then by Cauchy's integral formula

  $$\int_\gamma \frac{e^z}{z}\, dz = 2\pi i e^0 \neq 0.$$

  Therefore by the 'path independence theorem' (*c.f.* Theorem 2.1.9 in the text) g does not have an antiderivative on B.