Complex Analysis
Math 185A, Winter 2010
Sample Final Exam Questions

1. (a) Consider the change of variables from (x, y) to $(z, \bar{z})$ given by

$$z = x + iy, \qquad \bar{z} = x - iy. \tag{1}$$

If ∂_z denotes the partial derivative with respect to z keeping $\bar{z}$ fixed and $\partial_{\bar{z}}$ denotes the partial derivative with respect to $\bar{z}$ keeping z fixed, show that

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right),$$

where ∂_x denotes the partial derivative with respect to x keeping y fixed and ∂_y denotes the partial derivative with respect to y keeping x fixed.
(b) Write a complex valued function $f : \mathbb{C} \to \mathbb{C}$ as

$$f(z, \bar{z}) = u(x, y) + iv(x, y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ and $(z, \bar{z})$ are given in terms of (x, y) by (1). Show that f is analytic if and only if $\partial_{\bar{z}} f = 0$.

2. Let the simple closed curve γ be the positively oriented unit square with corners at $z = 0$, $z = 1$, $z = 1 + i$, $z = i$. Evaluate the following contour integrals:

$$\text{(a)} \int_\gamma \bar{z} dz; \qquad \text{(b)} \int_\gamma \frac{e^z}{z^2 - z + 1}\, dz.$$

3. Find the radius of convergence of the following power series.

$$\text{(a)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n; \quad \text{(b)} \sum_{n=0}^{\infty} \frac{3^n}{1+2^n} z^n; \quad \text{(c)} \sum_{n=0}^{\infty} \frac{1}{2^n} z^{3n}.$$

4. Let γ be the circle of radius 3 center the origin. Evaluate

$$\int_\gamma \frac{e^z}{z^3 + 4z^2 + 4z}\, dz.$$

5. Prove that the function

$$
\begin{aligned}
f(z) &= 1 + \sum_{n=1}^{\infty} \frac{1}{4n(4n-1)(4n-4)(4n-5)\cdots 4\cdot 3} z^{4n} \\
&= 1 + \frac{1}{4\cdot 3} z^4 + \frac{1}{8\cdot 7\cdot 4\cdot 3} z^8 + \frac{1}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} z^{12} + \dots
\end{aligned}
$$

is a solution of the differential equation

$$f'' = z^2 f$$

for every $z \in \mathbb{C}$.

6. Find the image of the quarter plane

$$\Omega = \{z \in \mathbb{C} : 0 < |z| < \infty,\ 0 < \arg z < \pi/2\}$$

under the mapping $w = e^{iz^2}$. Is this a conformal map on Ω? Is it one-to-one?