1. [15%] Suppose that \((a_n)\) is a sequence of positive real numbers, and let

\[ r = \limsup_{n \to \infty} a_n^{1/n}. \]

Prove that \(\sum_{n=1}^\infty a_n\) converges if \(r < 1\) and diverges if \(r > 1\). Give examples to show that a series may converge or diverge if \(r = 1\).

2. [15%] If \(A, B\) are subsets of a normed linear space \(X\), define

\[ A + B = \{ x \in X \mid x = a + b \text{ for } a \in A \text{ and } b \in B \}. \]

(a) Show that \(A + B\) is open if \(A\) is open.
(b) Show that \(A + B\) is closed if \(A\) is compact and \(B\) is closed.
(c) Give an example to show that \(A + B\) need not be closed if \(A\) and \(B\) are closed.

3. [10%] Let

\[ F = \{ f \in C([0,1]) \mid f(0) = f(1) = 0, \]

\[ |f(x) - f(y)| \leq |x - y|^{1/3} \text{ for all } x, y \in [0,1] \}. \]

Prove that \(F\) is a compact subset of \(C([0,1])\) (equipped with the sup-norm).

4. [15%] Let \(c_0\) be the Banach space of real sequences \(x = (x_n)\) such that \(x_n \to 0\) as \(n \to \infty\), and \(c\) the Banach space of convergent real sequences, both equipped with the sup-norm \(\|x\| = \sup_{n \in \mathbb{N}} |x_n|\).

(a) For any \(x \in c_0\) with \(\|x\| = 1\), show that there exist \(y, z \in c_0\) such that \(y \neq z\), \(\|y\| = \|z\| = 1\), and

\[ x = \frac{1}{2}(y + z). \]

(b) Define what it means for two normed linear spaces to be isometrically isomorphic. Deduce from (a) that \(c_0\) and \(c\) are not isometrically isomorphic. (We saw in class that they are topologically isomorphic.)
5. [15%] Suppose that $A \in B(X)$ a bounded linear operator on a Banach space $X$.

(a) Show that for any $t \in \mathbb{R}$ the series

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

converges with respect to the operator norm in $B(X)$, thus defining a function $f : \mathbb{R} \to B(X)$.

(b) Prove that $f$ is differentiable, in the sense that the limit

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists with respect to the operator norm convergence on $B(X)$ for each $t \in \mathbb{R}$, and compute the derivative $f' : \mathbb{R} \to B(X)$.

6. [15%] Show that if $|\lambda| < 1$, the following nonlinear integral equation

$$f(x) + \lambda \int_{-\infty}^{\infty} \frac{\cos [yf(y)]}{(1 + x^2 + y^2)^2} \, dy = 0$$

has a unique continuous solution $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.

7. [15%] For $n \in \mathbb{N}$, define $I_n : C([0, 1]) \to \mathbb{R}$ by

$$I_n(f) = \int_0^1 f(x) \cos (ne^x) \, dx.$$ 

(a) Show that if $p$ is a polynomial, then $I_n(p) \to 0$ as $n \to \infty$. **Hint:** Integration by parts.

(b) Show that $I_n(f) \to 0$ as $n \to \infty$ for any $f \in C([0, 1])$.

(c) Show that $I_n \in C([0, 1])^*$, where $C([0, 1])$ is equipped with the sup-norm. Does the sequence $(I_n)$ converge with respect to the weak-* topology on $C([0, 1])^*$? If so, what is its limit?