Analysis
Math 201A, Fall 2004
Final

Closed Book. Give complete proofs.
You can use any standard theorem, provided you state it carefully.

1. [15%] Suppose that (a_n) is a sequence of positive real numbers, and let

$$r = \limsup_{n\to\infty} a_n^{1/n}.$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$. Give examples to show that a series may converge or diverge if $r = 1$.

2. [15%] If A, B are subsets of a normed linear space X, define

$$A + B = \{x \in X \mid x = a + b \text{ for } a \in A \text{ and } b \in B\}.$$

(a) Show that $A + B$ is open if A is open.

(b) Show that $A + B$ is closed if A is compact and B is closed.

(c) Give an example to show that $A + B$ need not be closed if A and B are closed.

3. [10%] Let

$$\begin{aligned}\mathcal{F} &= \{f \in C([0,1]) \mid f(0) = f(1) = 0, \\ &\qquad |f(x) - f(y)| \le |x - y|^{1/3} \text{ for all } x, y \in [0,1]\}.\end{aligned}$$

Prove that $\mathcal{F}$ is a compact subset of $C([0,1])$ (equipped with the sup-norm).

4. [15%] Let c_0 be the Banach space of real sequences $x = (x_n)$ such that $x_n \to 0$ as $n \to \infty$, and c the Banach space of convergent real sequences, both equipped with the sup-norm $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$.

(a) For any $x \in c_0$ with $\|x\| = 1$, show that there exist $y, z \in c_0$ such that $y \neq z$, $\|y\| = \|z\| = 1$, and

$$x = \frac{1}{2}(y + z).$$

(b) Define what it means for two normed linear spaces to be isometrically isomorphic. Deduce from (a) that c_0 and c are not isometrically isomorphic. (We saw in class that they are topologically isomorphic.)

5. [15%] Suppose that $A \in \mathcal{B}(X)$ a bounded linear operator on a Banach space X.

(a) Show that for any $t \in \mathbb{R}$ the series

$$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

converges with respect to the operator norm in $\mathcal{B}(X)$, thus defining a function $f : \mathbb{R} \to \mathcal{B}(X)$.

(b) Prove that f is differentiable, in the sense that the limit

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists with respect to the operator norm convergence on $\mathcal{B}(X)$ for each $t \in \mathbb{R}$, and compute the derivative $f' : \mathbb{R} \to \mathcal{B}(X)$.

6. [15%] Show that if $|\lambda| < 1$, the following nonlinear integral equation

$$f(x) + \lambda \int_{-\infty}^{\infty} \frac{\cos\left[y f(y)\right]}{(1 + x^2 + y^2)^2}\, dy = 0$$

has a unique continuous solution $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.

7. [15%] For $n \in \mathbb{N}$, define $I_n : C([0,1]) \to \mathbb{R}$ by

$$I_n(f) = \int_0^1 f(x) \cos\left(n e^x\right)\, dx.$$

(a) Show that if p is a polynomial, then $I_n(p) \to 0$ as $n \to \infty$. HINT: Integration by parts.

(b) Show that $I_n(f) \to 0$ as $n \to \infty$ for any $f \in C([0,1])$.

(c) Show that $I_n \in C([0,1])^*$, where $C([0,1])$ is equipped with the sup-norm. Does the sequence (I_n) converge with respect to the weak-$*$ topology on $C([0,1])^*$? If so, what is its limit?