Final Exam, Tuesday, December 9, 2003

This exam has 9 numbered pages. Show your work and explain your reasoning. If you use a theorem from the textbook, either refer to it by name or give its statement.

Your Name:

Problem	Max.	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1. Let $f(x) = \sqrt{x}$ be defined on $[0, +\infty)$ with the usual metric.

a) (5 points) Is f Lipshitz continuous? Prove or disprove.

b) (5 points) Is f uniformly continuous? Prove or disprove.

c) (5 points) Now consider f restricted to the interval $[0, 1]$, and suppose (p_n) is a sequence of polynomial functions converging uniformly to f. Is the set $A = \{p_n \mid n \geq 1\} \cup \{f\}$ equicontinuous? Prove or disprove.

d) (5 points) Let C_n be the Lipshitz constants of the polynomials p_n of the previous part. Show that $C_n < \infty$, for all n, and $\lim_n C_n = +\infty$.

Problem 2. Let X be a Banach space.

a) (15 points) Let $B \subset X$ be a bounded closed set and $C \subset X$ a compact set, both non-empty. Define $B + C \equiv A = \{x + y \mid x \in B, y \in C\}$. Prove that A is compact iff B is compact.

b) (5 points) Give an example of a pair of non-empty sets $B, C \subset \mathbb{R}$, such that B is bounded and C is compact, $B + C$ is compact but B is *not* compact.

Problem 3. Let X be a Banach space over $\mathbb{C}$, and (A_n) a sequence in $\mathcal{B}(X)$.

a) (10 points) Prove that if A_n is compact for all n, and $A_n \to A$ uniformly, then A is compact.

b) (10 points) Consider the operator K defined on $C([0,1])$ by

$$Kf(x) = \int_0^1 k(xy)f(y)\,dy$$

where $k \in C([0,1])$. Prove that K is a compact operator on $C([0,1])$ considered with the supremum norm.

Problem 4. Consider $A = \{0, 1\}$ with the discrete topology, and define X as the cartesian product of a countable number of copies of A, i.e.,

$$X = \{(b_n)_{n \geq 0} \mid b_n \in A, n \geq 0\}$$

a) (5 points) Give a base for the product topology on X.

b) (5 points) Is the product topology on X metric? If so, give a metric that produces it. If not, explain why not.

c) (10 points) Show that a function $f : X \to \mathbb{C}$ is continuous iff

$$\lim_n \sup\{|f(b) - f(b')| \mid b, b' \in X, b_k = b'_k, k = 0, \dots, n\} = 0$$

Problem 5. True or False?

a) (2 points) Every finite-dimensional complex Banach space is isometrically isomorphic to $\mathbb{C}^n$, for some integer $n \geq 0$.

☐ True. ☐ False.

b) (2 points) If a bounded linear transformation T on a Banach space X is one-to-one and onto, then it has unique a bounded inverse

☐ True. ☐ False.

c) (2 points) For every $f \in C(\mathbb{R})$ there is a sequence of polynomials (p_n), such that $p_n \to f$ uniformly.

☐ True. ☐ False.

d) (2 points) If a sequence (A_n) in $\mathcal{B}(C([0,1]))$ converges in the strong operator topology to $A \in \mathcal{B}(C([0,1]))$, then it converges in the operator norm topology.

☐ True. ☐ False.

e) (2 points) Let X be a linear space equipped with a metric d. Then, $\|x\| = d(0,x)$, for all $x \in X$, defines a norm on X.

☐ True. ☐ False.

f) (2 points) For any convergent sequence, (f_n), in $C([0,1])$ with the sup norm, the set $\{f_n \mid n \geq 1\}$ is equicontinuous.

☐ True. ☐ False.

g) (2 points) For any two normed linear spaces, the space $\mathcal{B}(X,Y)$ considered with the operator norm is a Banach space.

☐ True. ☐ False.

h) (2 points) Every linear functional on $\mathcal{B}(X)$, for some Banach space X, which is continuous for the strong operator topology on $\mathcal{B}(X)$, is also continuous for the operator norm.

☐ True. ☐ False.

i) (2 points) Let X be a metric space and $f : X \to \mathbb{R}$ a lower semi-continuous function. If X is compact, then f attains its infimum.

☐ True. ☐ False.

j) (2 points) The IVP (for a real-valued function u) given by

$$\dot{u}(t) = |u(t)|^{1/3}$$
$$u(1) = 1$$

has a unique solution defined for $t \in (0,2)$

☐ True. ☐ False.