Problem Set 1
Math 201A, Fall 2006
Due: Friday, Oct 6

Problem 1. Give an ϵ-δ proof that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$.

Problem 2. If x, y, z are points in a metric space (X, d), show that

$$\begin{aligned} d(x,y) &\geq |d(x,z) - d(y,z)|, \\ d(x,y) + d(z,w) &\geq |d(x,z) - d(y,w)|. \end{aligned}$$

Prove that if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x,y)$.

Problem 3. If (X, d_X) and (Y, d_Y) are metric spaces, show that $d = d_X \times d_Y$ defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, is a metric on the Cartesian product $Z = X \times Y$.
If $X = Y = \mathbb{R}$ and $d_X(x,y) = d_Y(x,y) = |x - y|$, describe the set

$$\left\{ z \in \mathbb{R}^2 \mid d(z, 0) < 1 \right\}.$$

Problem 4. If X is a normed linear space with norm $\|\cdot\|$, define $\rho : X \to \mathbb{R}$ by

$$\rho(x) = \frac{\|x\|}{1+\|x\|}.$$

(a) Why isn't ρ a norm on X?

(b) Define $r : X \times X \to \mathbb{R}$ by

$$r(x,y) = \rho(x-y).$$

Prove that r is a metric on X.

(c) Define the diameter of X with respect to a metric d by

$$\operatorname{diam}(X) = \sup_{x,y\in X} d(x,y).$$

What is the diameter of X with respect to the metric $d(x,y) = \|x-y\|$? What is the diameter of X with respect to the metric $r(x,y) = \rho(x-y)$?

(d) Prove that $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $r(x_n, x) \to 0$ as $n \to \infty$.

Problem 5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers, and define

$$d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$$

by

$$d_1(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|, \qquad d_2(n,m) = |n-m|.$$

(a) Prove that d_1, d_2 are metrics on $\mathbb{N}$.

(b) Determine whether or not $\mathbb{N}$ is complete with respect each of the metrics d_1, d_2.