Problem Set 2
Math 201A, Fall 2006
Due: Friday, Oct 13

Problem 1. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series in a Banach space X such that $\|x_n\| \leq a_n$. If $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{R}$, prove that $\sum_{n=1}^{\infty} x_n$ converges in X.

Problem 2. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and define

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1},$$

with the obvious conventions for $R = 0$ and $R = \infty$. Prove that the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

for $x \in \mathbb{R}$ converges if $|x| < R$ and diverges if $|x| > R$.
Hint. Consider $|x| \leq (1-\epsilon)R$ and $|x| \geq (1+\epsilon)R$ for any $\epsilon > 0$.

Problem 3. Consider a set of real numbers,

$$\{x_{n,\alpha} \mid n \in \mathbb{N}, \alpha \in \mathcal{A}\},$$

indexed by $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary set.
(a) Prove that

$$\limsup_{n\to\infty} \left(\inf_{\alpha\in\mathcal{A}} x_{n,\alpha} \right) \leq \inf_{\alpha\in\mathcal{A}} \left(\limsup_{n\to\infty} x_{n,\alpha} \right). \tag{1}$$

Hint. Show that the number on the left is a lower bound of the set on the right.
(b) By changing $x_{n,\alpha} \mapsto -x_{n,\alpha}$, deduce the corresponding inequality for lim inf and sup.
(c) Give an example to show that there may be strict inequality in (1).

Problem 4. Suppose that F and G are, respectively, closed and open subsets of a metric space (X, d) such that $F \subset G$. Show that there is a continuous function $f : X \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$, $f(x) = 1$ for $x \in F$, and $f(x) = 0$ for $x \in G^c$.

Hint. Consider the function

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}.$$

This result is called *Urysohn's lemma.*

Problem 5. A metric space (X, d) is said to be an ultrametric space if

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \qquad \text{for all } x, y, z \in X.$$

Prove that in an ultrametric space, every open ball

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

is also closed.