Problem Set 3
Math 201A, Fall 2006
Due: Friday, October 20

Problem 1. Let X be the space of all real sequences of the form

$$x = (x_1, x_2, x_3, \ldots, x_n, 0, 0, \ldots), \qquad x_i \in \mathbb{R}$$

whose terms are zero from some point on. Define

$$\|x\|_\infty = \max_{i \in \mathbb{N}} |x_i|.$$

(a) Show that $(X, \|\cdot\|_\infty)$ is a normed linear space.

(b) Show that X is not complete.

(c) Give a description of the completion of X as a space of sequences.

Problem 2. Suppose that (X, d_X) and (Y, d_Y) are metric spaces and (Y, d_Y) is complete. If D is a dense subset of X and $f : D \to Y$ is uniformly continuous on D, prove that there exists a unique continuous function $F : X \to Y$ such that $F|_D = f$.

Problem 3. Fix a prime number p. For any nonzero rational number $r \in \mathbb{Q}$ there is a unique integer $k \in \mathbb{Z}$ such that $r = mp^k/n$, where m, n are integers that are not divisible by p. We then define $|r|_p = p^{-k}$. We define $|0|_p = 0$.

(a) Prove that $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ satisfies:

1. $|r|_p \geq 0$ and $|r|_p = 0$ if and only if $r = 0$;
2. $|-r|_p = |r_p|$;
3. $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

Deduce that $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by

$$d(r, s) = |r - s|_p$$

is an ultrametric on $\mathbb{Q}$. Show that $(\mathbb{Q}, d)$ is not complete.

(b) Let $(\mathbb{Q}_p, d_p)$ denote the completion of $(\mathbb{Q}, d)$. Use the result of Problem 2 to prove that addition $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ extends to a unique continuous function $+_p : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$.

Remark. Elements of $\mathbb{Q}_p$ are called p-adic numbers, which are important in algebraic number theory.

Problem 4. A metric space is said to be: *connected* if it is not the union of two disjoint non-empty open sets; *totally disconnected* if the only non-empty connected subspaces consist of a single point; and *perfect* if every point in the space is an accumulation point, meaning that it is a limit of a sequence of other points in the space.

Let $X = \{0,1\}^{\mathbb{N}}$ be the space of all sequences consisting of zeros or ones:

$$X = \{(s_1, s_2, s_3, \ldots) \mid s_n \in \{0,1\}\}.$$

Define $d : X \times X \to \mathbb{R}$ by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}$$

where $\mathbf{s} = (s_1, s_2, s_3, \ldots)$, $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, and

$$\delta_n = \begin{cases} 0 & \text{if } s_n = t_n, \\ 1 & \text{if } s_n \neq t_n. \end{cases}$$

(a) Prove that d is a metric on X.

(b) Prove that X is compact, totally disconnected, and perfect.

(c) Prove that the Cantor set C, regarded as a metric subspace of $[0,1]$ with the standard metric, is homeomorphic to X. (You can assume that the Cantor set is in one-to-one correspondence with the set of numbers that have a base-three expansion $0.b_1b_2b_3\ldots$ with no 1's, and that for any such number this base-three expansion is unique.)

(d) Define the shift map $\sigma : X \to X$ by

$$\sigma(s_1, s_2, s_3, \ldots) = (s_2, s_3, s_4, \ldots).$$

Prove that σ is continuous.

(e) Let $\sigma^n = \sigma \circ \sigma \circ \ldots \circ \sigma$ denote the n-fold composition of σ with itself. Show that there exists a $\delta > 0$ such that for any $\mathbf{s} \in X$ and any neighborhood U of $\mathbf{s}$, there exists $\mathbf{t} \in U$ and $n \in \mathbb{N}$ with

$$d\left(\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})\right) > \delta.$$

(e) Prove that there is a point $\mathbf{s} \in X$ such that the orbit of $\mathbf{s}$ under σ,

$$\{\sigma^n(\mathbf{s}) \mid n = 0, 1, 2, \ldots\},$$

is dense in X.