Problem Set 4
Math 201A, Fall 2006
Due: Friday, October 27

Problem 1. Give an example of a sequence (f_n) of continuous functions $f_n : [0, 1] \to \mathbb{R}$ that converges to 0 with respect to the L^1-norm,

$$\|f\|_1 = \int_0^1 |f(x)|\, dx,$$

such that the real sequence of pointwise values $(f_n(x))$ does not coverge for any $0 \le x \le 1$. Verify that there is a subsequence that converges pointwise-a.e. to 0.

Problem 2. The sequence space ℓ^∞ is the Banach space of all bounded real sequences,

$$\ell^\infty = \{(x_1, x_2, \dots, x_n, \dots) \mid x_n \in \mathbb{R},\ \exists M \in \mathbb{R} \text{ s.t. } |x_n| \le M \text{ for all } n \in \mathbb{N}\},$$

with the norm

$$\|(x_1, x_2, \dots, x_n, \dots)\| = \sup_{n\in\mathbb{N}} |x_n|.$$

Let

$$B = \{(x_1, x_2, \dots, x_n, \dots) \mid 0 \le x_n \le 1 \text{ for all } n \in \mathbb{N}\}.$$

Show that B is a closed, bounded subset of ℓ^∞ that is not compact. (You don't need to verify that ℓ^∞ is a Banach space.)

Problem 3. Suppose that (x_n) is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.

Problem 4. (a) Prove that a real-valued function $f : X \to \mathbb{R}$ is sequentially lower semicontinuous on X if and only if for every $a \in \mathbb{R}$ the set $f^{-1}((a, \infty))$ is open in X.

(b) If X is a compact metric space and $f : X \to \mathbb{R}$ is sequentially lower semicontinuous on X, prove that f is bounded from below and attains its minimum value. Give examples to show that such an f need not be bounded from above and need not attain its supremum even if it is.