ANALYSIS
Math 201A, Fall 2006
Problem Set 5

1. Suppose that (x_n) is a bounded sequence of real numbers. Define a sequence (y_n) by

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

(a) Prove that

$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} x_n.$$

(b) If (x_n) converges, must (y_n) converge? If (y_n) converges, must (x_n) converge? Prove your answers.

2. Let A be a subset of a metric space X. Define the characteristic function $\chi_A : X \to \mathbb{R}$ of A by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that χ_A is lower semi-continuous if and only if A is open.

3. Let $C_c(\mathbb{R})$ be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support, meaning that there exists an $R > 0$ (depending on f) such that $f(x) = 0$ for $|x| > R$. We define the sup-norm $\|\cdot\|_\infty$ and the L^1-norm $\|\cdot\|_1$ on $C_c(\mathbb{R})$ by

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|, \qquad \|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

(a) Show that $\|f\|_\infty$ and $\|f\|_1$ are finite for any $f \in C_c(\mathbb{R})$.

(b) Is $C_c(\mathbb{R})$ equipped with the sup-norm a Banach space? Prove your answer.

(c) Let (f_n) be a sequence in $C_c(\mathbb{R})$. Answer the following questions, and give a proof or counterexample.

1. If $f_n \to f \in C_c(\mathbb{R})$ as $n \to \infty$ with respect to the L^1-norm, does $f_n \to f$ as $n \to \infty$ with respect to the sup-norm?

2. If $f_n \to f \in C_c(\mathbb{R})$ as $n \to \infty$ with respect to the sup-norm, does $f_n \to f$ as $n \to \infty$ with respect to the L^1-norm?

4. A collection of sets has the *finite intersection property* if every finite subcollection has nonempty intersection.

(a) Prove that a metric space X is compact if and only if every collection of closed sets with the finite intersection property has non-empty intersection.

(b) Give an example of a collection of closed subsets of $(0, 1]$ (with its usual metric topology as a subset of $\mathbb{R}$) that has the finite intersection property but whose intersection is empty.

5. Let ℓ^∞ be the space of real, bounded sequences,

$$\ell^\infty = \{(x_1, x_2, x_3, \ldots) \mid x_n \in \mathbb{R}, \exists M > 0 \text{ s.t. } |x_n| \leq M \text{ for all } n \in \mathbb{N}\},$$

equipped with the sup-norm

$$\|(x_1, x_2, x_3, \ldots)\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Prove that the 'Hilbert cube'

$$C = \{(x_1, x_2, x_3, \ldots) \mid 0 \leq x_n \leq 1/n \text{ for every } n \in \mathbb{N}\}$$

is a compact subset of ℓ^∞. (You can assume that ℓ^∞ is a Banach space.)