1. Suppose that \((x_n)\) is a bounded sequence of real numbers. Define a sequence \((y_n)\) by
\[
y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}.
\]
(a) Prove that
\[
\liminf_{n \to \infty} x_n \leq \liminf_{n \to \infty} y_n \leq \limsup_{n \to \infty} y_n \leq \limsup_{n \to \infty} x_n.
\]
(b) If \((x_n)\) converges, must \((y_n)\) converge? If \((y_n)\) converges, must \((x_n)\) converge? Prove your answers.

2. Let \(A\) be a subset of a metric space \(X\). Define the characteristic function \(\chi_A : X \to \mathbb{R}\) of \(A\) by
\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{if } x \notin A.
\end{cases}
\]
Prove that \(\chi_A\) is lower semi-continuous if and only if \(A\) is open.

3. Let \(C_c(\mathbb{R})\) be the space of continuous functions \(f : \mathbb{R} \to \mathbb{R}\) with compact support, meaning that there exists an \(R > 0\) (depending on \(f\)) such that \(f(x) = 0\) for \(|x| > R\). We define the sup-norm \(\| \cdot \|_\infty\) and the \(L^1\)-norm \(\| \cdot \|_1\) on \(C_c(\mathbb{R})\) by
\[
\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|, \quad \|f\|_1 = \int_{-\infty}^{\infty} |f(x)| \, dx.
\]
(a) Show that \(\|f\|_\infty\) and \(\|f\|_1\) are finite for any \(f \in C_c(\mathbb{R})\).
(b) Is \(C_c(\mathbb{R})\) equipped with the sup-norm a Banach space? Prove your answer.
(c) Let \((f_n)\) be a sequence in \(C_c(\mathbb{R})\). Answer the following questions, and give a proof or counterexample.

1. If \(f_n \to f \in C_c(\mathbb{R})\) as \(n \to \infty\) with respect to the \(L^1\)-norm, does \(f_n \to f\) as \(n \to \infty\) with respect to the sup-norm?

2. If \(f_n \to f \in C_c(\mathbb{R})\) as \(n \to \infty\) with respect to the sup-norm, does \(f_n \to f\) as \(n \to \infty\) with respect to the \(L^1\)-norm?
4. A collection of sets has the finite intersection property if every finite subcollection has nonempty intersection.
(a) Prove that a metric space $X$ is compact if and only if every collection of closed sets with the finite intersection property has non-empty intersection.
(b) Give an example of a collection of closed subsets of $(0, 1]$ (with its usual metric topology as a subset of $\mathbb{R}$) that has the finite intersection property but whose intersection is empty.

5. Let $\ell^\infty$ be the space of real, bounded sequences,
\[ \ell^\infty = \{(x_1, x_2, x_3, \ldots) \mid x_n \in \mathbb{R}, \exists M > 0 \text{ s.t. } |x_n| \leq M \text{ for all } n \in \mathbb{N} \}, \]
equipped with the sup-norm
\[ \|(x_1, x_2, x_3, \ldots)\|_\infty = \sup_{n \in \mathbb{N}} |x_n|. \]
Prove that the ‘Hilbert cube’
\[ C = \{(x_1, x_2, x_3, \ldots) \mid 0 \leq x_n \leq 1/n \text{ for every } n \in \mathbb{N} \}\]
is a compact subset of $\ell^\infty$. (You can assume that $\ell^\infty$ is a Banach space.)