1. Let \( C^1([0, 1]) \) denote the space of continuously differentiable functions \( f : [0, 1] \to \mathbb{R} \), and define
\[
\| f \| = \sup_{x \in [0, 1]} |f(x)| + \sup_{x \in [0, 1]} |f'(x)|.
\]
(a) Show that \( \| \cdot \| \) is a norm on \( C^1([0, 1]) \).
(b) Prove that \( C^1([0, 1]) \) is a Banach space with respect to \( \| \cdot \| \).

**Warning.** The uniform convergence of \( (f_n) \) to \( f \) does not imply the convergence of \( (f'_n) \) to \( f' \).

2. If \( f : [0, 1] \to \mathbb{R} \) is integrable, define \( b_n \in \mathbb{R} \) by
\[
b_n = \int_0^1 f(x) \sin(n\pi x) \, dx.
\]
(a) Prove that \( b_n \to 0 \) as \( n \to \infty \) for any polynomial.
(b) Prove that \( b_n \to 0 \) as \( n \to \infty \) for any \( f \in C([0, 1]) \).

**HINT.** Integrate by parts for (a), and use the Weierstrass approximation theorem for (b).

3. A function \( f : [0, 1] \to \mathbb{R} \) is said to be Hölder continuous with exponent \( \alpha \) if
\[
[f]_{\alpha} = \sup_{x \neq y \in [0, 1]} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}
\]
is finite. Given \( 0 < \alpha \leq 1 \) and \( M > 0 \), define
\[
\mathcal{F} = \{ f \in C([0, 1]) \mid \| f \|_{\infty} \leq M, \ [f]_{\alpha} \leq M \}.
\]
Prove that \( \mathcal{F} \) is a compact subset of \( C([0, 1]) \) equipped with the sup-norm \( \| \cdot \|_{\infty} \).
4. Suppose that \((f_n)\) is a sequence of continuous functions \(f_n : [0, 1] \to \mathbb{R}\) such that \(|f_n(x)| \leq 1\) for all \(n \in \mathbb{N}, x \in [0, 1]\). Define \(F_n : [0, 1] \to \mathbb{R}\) by
\[
F_n(x) = \int_0^x f_n(t) \, dt.
\]
Prove that the sequence \((F_n)\) has a subsequence that converges uniformly on \([0, 1]\).

5. Suppose that
\[
\{f_n : K \to \mathbb{R} \mid n \in \mathbb{N}\}
\]
is an equicontinuous family of functions on a compact metric space \(K\). If \((f_n)\) converges pointwise to a function \(f\), prove that \(f\) is continuous. Is the convergence necessarily uniform?