Analysis
**Math 201A, Fall 2006**
**Problem Set 6**

**1.** Let $C^1([0,1])$ denote the space of continuously differentiable functions $f : [0,1] \to \mathbb{R}$, and define

$$\|f\| = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| .$$

(a) Show that $\|\cdot\|$ is a norm on $C^1([0,1])$.

(b) Prove that $C^1([0,1])$ is a Banach space with respect to $\|\cdot\|$.

Warning. The uniform convergence of $(f_n)$ to $f$ does not imply the convergence of $(f_n')$ to $f'$.

**2.** If $f : [0,1] \to \mathbb{R}$ is integrable, define $b_n \in \mathbb{R}$ by

$$b_n = \int_0^1 f(x) \sin(n\pi x)\, dx.$$

(a) Prove that $b_n \to 0$ as $n \to \infty$ for any polynomial.

(b) Prove that $b_n \to 0$ as $n \to \infty$ for any $f \in C([0,1])$.

Hint. Integrate by parts for (a), and use the Weierstrass approximation theorem for (b).

**3.** A function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous with exponent $\alpha$ if

$$[f]_\alpha = \sup_{x\neq y\in[0,1]} \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \right\}$$

is finite. Given $0 < \alpha \le 1$ and $M > 0$, define

$$\mathcal{F} = \{ f \in C([0,1]) \mid \|f\|_\infty \le M, \quad [f]_\alpha \le M \} .$$

Prove that $\mathcal{F}$ is a compact subset of $C([0,1])$ equipped with the sup-norm $\|\cdot\|_\infty$.

**4.** Suppose that $(f_n)$ is a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}$, $x \in [0,1]$. Define $F_n : [0,1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^x f_n(t)\, dt.$$

Prove that the sequence $(F_n)$ has a subsequence that converges uniformly on $[0,1]$.

**5.** Suppose that

$$\{f_n : K \to \mathbb{R} \mid n \in \mathbb{N}\}$$

is an equicontinuous family of functions on a compact metric space $K$. If $(f_n)$ converges pointwise to a function $f$, prove that $f$ is continuous. Is the convergence necessarily uniform?