Problem Set 7
Math 201A, Fall 2006
Due: Wednesday, November 22

Problem 1. Let $r_n = x_{n+1}/x_n$ be the ratio of successive terms in the Fibonacci sequence (x_n) defined by $x_{n+1} = x_n + x_{n-1}$ with $x_0 = x_1 = 1$. Prove that $r_n \to \phi$ as $n \to \infty$ where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

Problem 2. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = 1 + \log(1 + e^x)$$

satisfies

$$|Tx - Ty| < |x - y| \qquad \text{for all } x, y \in \mathbb{R} \text{ with } x \neq y,$$

but T does not have any fixed points. Why doesn't this example contradict the contraction mapping theorem?

Problem 3. Suppose that X is a compact metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) < d(x, y) \qquad \text{for all } x, y \in X \text{ with } x \neq y.$$

Prove that T has a unique fixed point in X.

Problem 4. Consider the following nonlinear integral equation:

$$f(x) - \frac{1}{\pi}\int_0^1 \frac{f^2(y)}{1 + x^2 + y^2}\, dy = \frac{3}{4}, \qquad 0 \leq x \leq 1.$$

Prove that there is a unique continuous solution $f : [0, 1] \to \mathbb{R}$ of this equation with the property that $0 \leq f(x) \leq 1$ for all $0 \leq x \leq 1$.