Problem 1. Let \( r_n = \frac{x_{n+1}}{x_n} \) be the ratio of successive terms in the Fibonacci sequence \( (x_n) \) defined by \( x_{n+1} = x_n + x_{n-1} \) with \( x_0 = x_1 = 1 \). Prove that \( r_n \to \phi \) as \( n \to \infty \) where \( \phi = \frac{1 + \sqrt{5}}{2} \) is the golden ratio.

Problem 2. Show that the mapping \( T: \mathbb{R} \to \mathbb{R} \) defined by
\[
T x = 1 + \log (1 + e^x)
\]
satisfies
\[
|T x - T y| < |x - y| \quad \text{for all } x, y \in \mathbb{R} \text{ with } x \neq y,
\]
but \( T \) does not have any fixed points. Why doesn’t this example contradict the contraction mapping theorem?

Problem 3. Suppose that \( X \) is a compact metric space and \( T: X \to X \) satisfies
\[
d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \neq y.
\]
Prove that \( T \) has a unique fixed point in \( X \).

Problem 4. Consider the following nonlinear integral equation:
\[
f(x) - \frac{1}{\pi} \int_0^1 \frac{f^2(y)}{1 + x^2 + y^2} \, dy = \frac{3}{4}, \quad 0 \leq x \leq 1.
\]
Prove that there is a unique continuous solution \( f: [0, 1] \to \mathbb{R} \) of this equation with the property that \( 0 \leq f(x) \leq 1 \) for all \( 0 \leq x \leq 1 \).