Problem 1. Recall that a function \( f : [0, 1] \to \mathbb{R} \) is Lipschitz continuous if its Lipschitz constant
\[
\text{Lip}(f) = \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}
\]
is finite.
(a) For \( M > 0 \), let
\[
L_M = \{ f \in C([0,1]) \mid \text{Lip}(f) \leq M \}.
\]
Show that \( L_M \) is a closed subset of \( C([0,1]) \) equipped with the sup-norm.
(b) Let \( L = \{ f \in C([0,1]) \mid f \text{ is Lipschitz continuous} \} \). Prove that \( L \) is a linear subspace of \( C([0,1]) \).
(c) Is \( L \) a closed linear subspace of \( C([0,1]) \) equipped with the sup-norm?

Problem 2. Let \( c_0(\mathbb{N}) \) be the Banach space of real sequences \( x = (x_1, x_2, x_3, \ldots) \) such that \( x_n \to 0 \) as \( n \to \infty \), equipped with the sup norm
\[
\|x\| = \sup_{n \in \mathbb{N}} |x_n|.
\]
(a) Let \( e_n = (0, 0, \ldots, 0, 1, 0, \ldots) \) be the sequence with \( n \)th term equal to 1 and all other terms equal to 0. Show that \( (e_n)_{n=1}^{\infty} \) is a Schauder basis of \( c_0(\mathbb{N}) \).
(b) Let
\[
f_1 = \frac{1}{2}e_1, \quad f_n = \frac{1}{2}e_n - e_{n-1} \quad \text{for } n \geq 2.
\]
Show that \( \{ f_n \mid n \in \mathbb{N} \} \) is a linearly independent set. Is \( (f_n)_{n=1}^{\infty} \) a Schauder basis of \( c_0(\mathbb{N}) \)?
Problem 3. Suppose that $X$, $Y$, $Z$ are normed linear spaces and $A : X \to Y$, $B : Y \to Z$ are bounded linear operators. Prove that $BA : X \to Z$ is a bounded linear operator, and

$$\|BA\| \leq \|A\|\|B\|.$$ 

Give an example to show that this inequality may be strict.

Problem 4. Let $\delta : C([0, 1]) \to \mathbb{R}$ be the functional that evaluates a function at the origin, defined by $\delta(f) = f(0)$.

(a) Show that $\delta$ is a linear functional. What is the kernel of $\delta$? What is the range of $\delta$?

(b) If $C([0, 1])$ is equipped with the sup-norm

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|,$$

show that $\delta$ is bounded and compute its norm.

(c) If $C([0, 1])$ is equipped with the 1-norm

$$\|f\|_1 = \int_0^1 |f(x)| \, dx,$$

show that $\delta$ is unbounded.

(d) Give an example of a bounded linear functional $F : C([0, 1]) \to \mathbb{R}$ where $C([0, 1])$ is equipped with the 1-norm and compute its norm.