Problem Set 8
Math 201A, Fall 2006
Due: Friday, December 1

Problem 1. Recall that a function $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous if its Lipschitz constant

$$\mathrm{Lip}(f) = \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}$$

is finite.

(a) For $M > 0$, let

$$L_M = \{f \in C([0,1]) \mid \mathrm{Lip}(f) \leq M\}.$$

Show that L_M is a closed subset of $C([0,1])$ equipped with the sup-norm.

(b) Let $L = \{f \in C([0,1]) \mid f \text{ is Lipschitz continuous}\}$. Prove that L is a linear subspace of $C([0,1])$.

(c) Is L a closed linear subspace of $C([0,1])$ equipped with the sup-norm?

Problem 2. Let $c_0(\mathbb{N})$ be the Banach space of real sequences $x = (x_1, x_2, x_3, \ldots)$ such that $x_n \to 0$ as $n \to \infty$, equipped with the sup norm

$$\|x\| = \sup_{n \in \mathbb{N}} |x_n|.$$

(a) Let $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ be the sequence with nth term equal to 1 and all other terms equal to 0. Show that $(e_n)_{n=1}^{\infty}$ is a Schauder basis of $c_0(\mathbb{N})$.

(b) Let

$$f_1 = \frac{1}{2} e_1, \qquad f_n = \frac{1}{2} e_n - e_{n-1} \quad \text{for } n \geq 2.$$

Show that $\{f_n \mid n \in \mathbb{N}\}$ is a linearly independent set. Is $(f_n)_{n=1}^{\infty}$ a Schauder basis of $c_0(\mathbb{N})$?

Problem 3. Suppose that X, Y, Z are normed linear spaces and $A : X \to Y$, $B : Y \to Z$ are bounded linear operators. Prove that $BA : X \to Z$ is a bounded linear operator, and

$$\|BA\| \leq \|A\| \|B\|.$$

Give an example to show that this inequality may be strict.

Problem 4. Let $\delta : C([0, 1]) \to \mathbb{R}$ be the functional that evaluates a function at the origin, defined by $\delta(f) = f(0)$.

(a) Show that δ is a linear functional. What is the kernel of δ? What is the range of δ?

(b) If $C([0, 1])$ is equipped with the sup-norm

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|,$$

show that δ is bounded and compute its norm.

(c) If $C([0, 1])$ is equipped with the 1-norm

$$\|f\|_1 = \int_0^1 |f(x)| \, dx,$$

show that δ is unbounded.

(d) Give an example of a bounded linear functional $F : C([0, 1]) \to \mathbb{R}$ where $C([0, 1])$ is equipped with the 1-norm and compute its norm.