Problem Set 9
Math 201A, Fall 2006
Due: Friday, December 8

Problem 1. Let c be the Banach space of all convergent real sequences $(x_n)_{n=1}^{\infty}$, and c_0 the subspace of sequences that converge to 0, both equipped with the ∞-norm, $\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
(a) Define $L : c \to \mathbb{R}$ by $L(x_n) = \lim_{n\to\infty} x_n$. Prove that L is a bounded linear functional on c and compute its norm.
(b) For $x = (x_n) \in c$, define $Tx = y$ where $y = (y_n)_{n=1}^{\infty}$ is given by

$$y_1 = Lx, \qquad y_{n+1} = x_n - Lx \quad \text{for } n \geq 1.$$

Prove that $T : c \to c_0$ is a one-to-one, onto bounded linear map. (It follows from the open mapping theorem that T^{-1} is bounded, so c_0 and c are topologically isomorphic Banach spaces.)

Problem 2. A sequence (T_n) of bounded linear operators $T_n : X \to Y$ on normed linear spaces X, Y is said to converge strongly to $T : X \to Y$ if $T_n x \to Tx$ in norm in Y for every $x \in X$.
(a) Show that if $T_n \to T$ uniformly (i.e. with respect to the operator norm), then $T_n \to T$ strongly.
(b) Let $C_0(\mathbb{R})$ be the Banach space of continuous functions that approach zero at ∞, equipped with the sup-norm. For $h \in \mathbb{R}$ define the translation operator $T_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$T_h f(x) = f(x+h).$$

Prove that $T_h \to I$ strongly as $h \to 0$, where I is the identity operator on $C_0(\mathbb{R})$. Prove that T_h does not converge to I uniformly as $h \to 0$. HINT. Any $f \in C_0(\mathbb{R})$ is uniformly continuous.
(c) With T_h as in (b), define $A_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$A_h = \frac{T_h - I}{h}.$$

Does A_h converge strongly as $h \to 0$? For what $f \in C_0(\mathbb{R})$ does $A_h f$ converge in norm as $h \to 0$? Compute the limit when it exists.

Problem 3. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with $m \times n$ matrix (a_{ij}) with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^m$.

(a) Compute the operator (or matrix) norm $\|A\|$ if the domain $\mathbb{R}^n$ is equipped with the 1-norm,

$$\|(x_1, \ldots, x_n)\|_1 = |x_1| + \ldots + |x_n|,$$

and the range $\mathbb{R}^m$ is equipped with the ∞-norm,

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

(b) If the domain $\mathbb{R}^n$ is equipped with the ∞-norm and the range $\mathbb{R}^m$ is equipped with the 1-norm, prove that

$$\|A\| \le \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|,$$

with equality if $a_{ij} \ge 0$ for all $1 \le i \le m$, $1 \le j \le n$.

Problem 4. Let $\ell^2(\mathbb{N})$ be the Banach space of square-sumable real sequences $x = (x_i)_{i=1}^\infty$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}.$$

A sequence $(x^{(n)})$ in $\ell^2(\mathbb{N})$,

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots),$$

converges weakly to $x = (x_1, x_2, x_3, \ldots)$ as $n \to \infty$ if for every

$$y = (y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N})$$

we have

$$\sum_{i=1}^{\infty} x_i^{(n)} y_i \to \sum_{i=1}^{\infty} x_i y_i \quad \text{as } n \to \infty.$$

Let $e^{(n)} = (0, 0, \ldots 0, 1, 0, \ldots)$ be the element of $\ell^2(\mathbb{N})$ with $x_i = 1$ when $i = n$ and $x_i = 0$ otherwise.

(a) Prove that the sequence $(e^{(n)})$ converges weakly to 0 as $n \to \infty$, but does not converge strongly (i.e. in norm) to any limit.

(b) Does the sequence $(ne^{(n)})$ converge weakly as $n \to \infty$?