Solutions: Problem Set 2
Math 201A, Fall 2006

Problem 1. Suppose that $\sum_{n=1}^{\infty} x_n$ is a series in a Banach space X such that $\|x_n\| \le a_n$. If $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{R}$, prove that $\sum_{n=1}^{\infty} x_n$ converges in X.

Solution.

- We denote the partial sums by

$$y_N = \sum_{n=1}^{N} x_n \in X, \qquad b_N = \sum_{n=1}^{N} a_n \in \mathbb{R}.$$

  For $M > N$, we have

$$\|y_M - y_N\| = \left\| \sum_{n=N+1}^{M} x_n \right\| \le \sum_{n=N+1}^{M} \|x_n\| \le \sum_{n=N+1}^{M} a_n = |b_M - b_N|.$$

  Since $\sum_{n=1}^{\infty} a_n$ converges, the sequence (b_N) is Cauchy in $\mathbb{R}$, and therefore (y_N) is Cauchy in X. Since X is complete, the sequence of partial sums (y_N), and hence the series $\sum_{n=1}^{\infty} x_n$, converges in X.

- For use in the next question, note that the convergence of $\sum_{n=1}^{\infty} x_n$ implies that $x_n \to 0$ as $n \to \infty$. This follows immediately from the Cauchy criterion for the partial sums and the fact that

$$x_n = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i.$$

  The converse is not true: a series may diverge even though $x_n \to 0$. (For example, the harmonic series $\sum_{n=1}^{\infty} 1/n$.)

- If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then we say that $\sum_{n=1}^{\infty} x_n$ converges absolutely. A series may converge without converging absolutely. (For example, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is convergent but not absolutely convergent.)

Problem 2. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and define

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1},$$

with the obvious conventions for $R = 0$ and $R = \infty$. Prove that the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

for $x \in \mathbb{R}$ converges if $|x| < R$ and diverges if $|x| > R$.

Solution.

- Suppose that $R = \infty$, meaning that

  $$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0.$$

  Then given any $x \neq 0$, there exists an N such that $n > N$ implies that $\sqrt[n]{|a_n|} < 1/(2|x|)$ or $|a_n x^n| < 2^{-n}$. Since $\sum_{n=0}^{\infty} 2^{-n}$ converges, the result of Problem 1 (with $X = \mathbb{R}$) implies that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$.

- Suppose that $R = 0$, meaning that

  $$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty.$$

  Then given any $x \neq 0$, there exist infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > 1/|x|$ and $|a_n x^n| > 1$. Hence, $a_n x^n$ does not converge to 0 as $n \to \infty$, and from Problem 1 the series does not converge. Thus, the series converges only for $x = 0$.

- Suppose that $0 < R < \infty$, meaning that

  $$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}.$$

- If $|x| < R$, then we can choose $0 < c < 1$ such that $|x|/c < R$. Since $c/|x| > 1/R$, it follows from the definition of the lim sup that there exists N such that $n > N$ implies $\sqrt[n]{|a_n|} < c/|x|$, and hence $|a_n x^n| < c^n$. Since $\sum_{n=0}^{\infty} c^n$ converges, Problem 1 implies that the power series converges absolutely.

- If $|x| > R$, then it follows from the definition of the $\limsup$ that there are infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > 1/|x|$, and $|a_n x^n| > 1$. Hence, $a_n x^n$ does not converge to 0 as $n \to \infty$, and the power series does not converge.

- The power series may converge or diverge at the endpoints $x = \pm R$.

Problem 3. Consider a set of real numbers,

$$\{x_{n,\alpha} \mid n \in \mathbb{N}, \alpha \in \mathcal{A}\},$$

indexed by $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is an arbitrary set.

(a) Prove that

$$\limsup_{n\to\infty} \left(\inf_{\alpha\in\mathcal{A}} x_{n,\alpha} \right) \leq \inf_{\alpha\in\mathcal{A}} \left(\limsup_{n\to\infty} x_{n,\alpha} \right). \tag{1}$$

(b) By changing $x_{n,\alpha} \mapsto -x_{n,\alpha}$, deduce the corresponding inequality for $\liminf$ and $\sup$.

(c) Give an example to show that there may be strict inequality in (1).

Solution.

- (a) Fix $n \in \mathbb{N}$. Since $\inf_{\alpha\in A} x_{n,\alpha}$ is a lower bound of the set $\{x_{n,\alpha} \mid \alpha \in A\}$, we have
$$\inf_{\alpha\in A} x_{n,\alpha} \leq x_{n,\beta}$$
for every $\beta \in A$. (If $\{x_{n,\alpha} \mid \alpha \in A\}$ is not bounded from below, then its infimum is $-\infty$ and the inequality also holds.) Taking the $\limsup_{n\to\infty}$ of this inequality, we get that for every $\beta \in A$
$$\limsup_{n\to\infty} \left(\inf_{\alpha\in A} x_{n,\alpha} \right) \leq \limsup_{n\to\infty} x_{n,\beta}.$$
This inequality implies that $\limsup_{n\to\infty} (\inf_{\alpha\in A} x_{n,\alpha})$ is a lower bound of the set $\{\limsup_{n\to\infty} x_{n,\beta} \mid \beta \in A\}$. Since the infimum of a set is the greatest upper bound, it follows that
$$\limsup_{n\to\infty} \left(\inf_{\alpha\in A} x_{n,\alpha} \right) \leq \inf_{\alpha\in A} \left(\limsup_{n\to\infty} x_{n,\alpha} \right).$$

- (b) The corresponding result for $\liminf$ and $\sup$ follows from an application of this inequality to the numbers $(-x_{n,\alpha})$:
$$\limsup_{n\to\infty} \left(\inf_{\alpha\in\mathcal{A}} (-x_{n,\alpha}) \right) \leq \inf_{\alpha\in\mathcal{A}} \left(\limsup_{n\to\infty} (-x_{n,\alpha}) \right),$$

which implies that

$$-\liminf_{n\to\infty}\left(\sup_{\alpha\in\mathcal{A}} x_{n,\alpha}\right) \le -\sup_{\alpha\in\mathcal{A}}\left(\liminf_{n\to\infty} x_{n,\alpha}\right).$$

Hence,

$$\sup_{\alpha}\left(\liminf_{n\to\infty} x_{n,\alpha}\right) \le \liminf_{n\to\infty}\left(\sup_{\alpha} x_{n,\alpha}\right).$$

- (c) Let $\mathcal{A} = \mathbb{R}$ (we could use $\mathcal{A} = \mathbb{N}$ equally well) and define

$$x_{n,\alpha} = \begin{cases} 0 & \text{if } n \le \alpha, \\ 1 & \text{if } n > \alpha. \end{cases}$$

Then for every $n \in \mathbb{N}$

$$\inf_{\alpha\in\mathbb{R}} x_{n,\alpha} = 0,$$

and hence

$$\limsup_{n\to\infty}\left(\inf_{\alpha\in\mathbb{R}} x_{n,\alpha}\right) = 0.$$

For every $\alpha \in \mathbb{R}$, the terms in the sequence $(x_{n,\alpha})$ are equal to 1 when n is large enough, so that

$$\limsup_{n\to\infty} x_{n,\alpha} = 1.$$

Thus,

$$\inf_{\alpha\in\mathbb{R}}\left(\limsup_{n\to\infty} x_{n,\alpha}\right) = 1,$$

and we have strict inequality in (1).

Problem 4. Suppose that F and G are, respectively, closed and open subsets of a metric space (X, d) such that $F \subset G$. Show that there is a continuous function $f : X \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$, $f(x) = 1$ for $x \in F$, and $f(x) = 0$ for $x \in G^c$.

Solution.

- We define the distance $d(x, A)$ of a point $x \in X$ from a set $A \subset X$ by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

  As proved in class, the function $x \mapsto d(x, A)$ is continuous for any $A \subset X$.

- We claim that $d(x, G^c) + d(x, F) > 0$ for every $x \in X$. Suppose not. Then

$$d(x, G^c) = d(x, F) = 0$$

  for some $x \in X$. It follows from the definition of the distance function that there exist sequences (x_n) in F and (y_n) in G^c such that $x_n \to x$ and $y_n \to x$ as $n \to \infty$. Since F and G^c are closed, $x \in F \cap G^c$, which contradicts the assumption that $F \subset G$.

- The function $f : X \to \mathbb{R}$ defined by

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}$$

  is a composition of continuous functions, since the denominator is never zero, so it is continuous. Since $d(x, A) = 0$ when $x \in A$, we see that f has the required properties.

Problem 5. A metric space (X, d) is said to be an ultrametric space if

$$d(x, y) \le \max\{d(x, z), d(z, y)\} \qquad \text{for all } x, y, z \in X.$$

Prove that in an ultrametric space, every open ball

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

is also closed.

Solution.

- Suppose that (y_n) is a sequence in $B_r(x)$, and $y_n \to y$ as $n \to \infty$. Then $d(y_n, y) < r$ for some $n \in \mathbb{N}$, since the sequence converges to y. Also $d(x, y_n) < r$ since $y_n \in B_r(x)$. Since d is an ultrametic, it follows that

  $$d(x, y) \le \max\{d(x, y_n), d(y_n, y)\} < r,$$

  so $y \in B_r(x)$, which proves that $B_r(x)$ is closed.