ANALYSIS
Solutions: Problem Set 3
Math 201A, Fall 2006

Problem 1. Let X be the space of all real sequences of the form

$$x = (x_1, x_2, x_3, \ldots, x_n, 0, 0, \ldots), \qquad x_i \in \mathbb{R}$$

whose terms are zero from some point on. Define

$$\|x\|_\infty = \max_{i \in \mathbb{N}} |x_i|.$$

(a) Show that $(X, \|\cdot\|_\infty)$ is a normed linear space.

(b) Show that X is not complete.

(c) Give a description of the completion of X as a space of sequences.

Solution.

- (a) X is a linear space under component-wise addition and scalar multiplication: if $\lambda \in \mathbb{R}$, $x = (x_i)$, $y = (y_i)$ then

  $$\lambda x = (\lambda x_i), \qquad x + y = (x_i + y_i).$$

  The 0-vector is the sequence $(0, 0, 0, \ldots)$.

- We have: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x_i = 0$ for every $i \in \mathbb{N}$, or $x = 0$; for any $\lambda \in \mathbb{R}$,

  $$\|\lambda x\|_\infty = \max_{i \in \mathbb{N}} |\lambda x_i| = |\lambda| \max_{i \in \mathbb{N}} |x_i| = |\lambda| \, \|x\|_\infty;$$

  and

  $$\begin{aligned} \|x + y\|_\infty &= \max_{i \in \mathbb{N}} |x_i + y_i| \\ &\leq \max_{i \in \mathbb{N}} \{|x_i| + |y_i|\} \\ &\leq \max_{i \in \mathbb{N}} |x_i| + \max_{i \in \mathbb{N}} |y_i| \\ &\leq \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

- (b) Define a sequence (x_n) in X by
$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right).$$
For $m > n$, we have
$$\|x_m - x_n\|_\infty = \frac{1}{n+1},$$
which implies that the sequence is Cauchy.

- If $y \in X$, then $y = (y_1, y_2, \dots, y_N, 0, 0 \dots)$ for some $N \in \mathbb{N}$. If $n > N$, then
$$\|x_n - y\|_\infty \geq \frac{1}{N+1},$$
so (x_n) does not converge to y as $n \to \infty$. Therefore, the sequence (x_n) has no limit in X, and X is not complete.

- (c) The completion of X is the space c_0 of all real sequences (x_i) such that $x_i \to 0$ as $i \to \infty$. The norm on c_0 is $\|\cdot\|_\infty$.

- The inclusion map $\phi : X \to c_0$, where $\phi(x) = x$, is an isometric imbedding of X into c_0, so by the uniqueness part of the completion theorem we just need to show that X is dense in c_0 and c_0 is complete.

- Given $y = (y_1, y_2, \dots, y_i, \dots) \in c_0$ let
$$x_n = (y_1, y_2, \dots, y_n, 0, 0 \dots) \in X.$$
Since $y_i \to 0$ as $i \to \infty$, we have
$$\|x_n - y\|_\infty = \max_{i>n} |y_i| \to 0 \qquad \text{as } n \to \infty.$$
Thus, every point in c_0 is a limit of a sequence in X, so X is dense in c_0.

- Suppose that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in c_0. Let $x_n = (x_{n,i})_{i=1}^\infty$. For each $i \in \mathbb{N}$ we have
$$|x_{n,i} - x_{m,i}| \leq \|x_n - x_m\|_\infty,$$
so $(x_{n,i})_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $y_i \in \mathbb{R}$ such that $x_{n,i} \to y_i$ as $n \to \infty$. Let $y = (y_i)_{i=1}^\infty$. We will show that $x_n \to y$ with respect to $\|\cdot\|_\infty$ and $y \in c_0$.

- Since (x_n) is Cauchy in c_0, given $\epsilon > 0$, there exists N depending only on ϵ such that

$$\|x_n - x_m\|_\infty < \epsilon \qquad \text{for all } n, m > N.$$

Hence for each $i \in \mathbb{N}$ and all $n, m > N$,

$$|x_{n,i} - x_{m,i}| < \epsilon.$$

Taking the limit of this inequality as $m \to \infty$, we get

$$|x_{n,i} - y_i| \leq \epsilon \qquad \text{for all } i \in \mathbb{N} \text{ and } n > N.$$

Taking the supremum of this inequality over i, we find that

$$\|x_n - y\|_\infty \leq \epsilon \qquad \text{for all } n > N.$$

Hence $\|x_n - y\|_\infty \to 0$ as $n \to \infty$.

- Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|x_N - y\|_\infty < \frac{\epsilon}{2}.$$

Since $x_{N,i} \to 0$ as $i \to \infty$, there exists $K \in \mathbb{N}$ such that

$$|x_{N,i}| < \frac{\epsilon}{2} \qquad \text{for } i > K.$$

It follows that

$$|y_i| \leq |y_i - x_{N,i}| + |x_{N,i}| < \epsilon \qquad \text{for } i > K.$$

Hence, $y_i \to 0$ as $i \to \infty$, so $y \in c_0$, and c_0 is complete.

Problem 2. Suppose that (X, d_X) and (Y, d_Y) are metric spaces and (Y, d_Y) is complete. If D is a dense subset of X and $f : D \to Y$ is uniformly continuous on D, prove that there exists a unique continuous function $F : X \to Y$ such that $F|_D = f$.

Solution.

- If $x \in X$ then there exists a sequence (a_n) in D such that $a_n \to x$ as $n \to \infty$, since D is dense in X. The sequence (a_n) is Cauchy in D, and the uniform continuity of f implies that $(f(a_n))$ is Cauchy in Y. Since Y is complete, there exists $y \in Y$ such that $f(a_n) \to y$ as $n \to \infty$. We define $F(x) = y$.

- This function $F : X \to Y$ is well-defined. If (a_n), (a'_n) are two sequences in D that converge to $x \in X$, then $d_X(a_n, a'_n) \to 0$ as $n \to \infty$, and the uniform continuity of f implies that

$$d_Y(f(a_n), f(a'_n)) \to 0 \qquad \text{as } n \to \infty.$$

  It follows that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(a'_n),$$

  so the value $F(x)$ is independent of the choice of sequence converging to x.

- If $a \in D$, then we may choose $a_n = a$ for every $n \in \mathbb{N}$, which gives $F(a) = f(a)$, so that the restriction of F to D is equal to f.

- To prove the continuity of F, suppose that $x, x' \in X$. Choose sequences (a_n), (a'_n) in D such that $a_n \to x$, $a'_n \to x'$. Given $\epsilon > 0$, the uniform continuity of f implies that there exists $\delta > 0$ such that $a, a' \in D$ and $d_X(a, a') < \delta$ implies that $d_Y(f(a), f(a')) < \epsilon/3$. Choose N such that

$$\begin{aligned} d(a_N, x) &< \frac{\delta}{3}, \qquad & d_Y(f(a_N), F(x)) &< \frac{\epsilon}{3}, \\ d(a'_N, x') &< \frac{\delta}{3}, \qquad & d_Y(f(a'_N), F(x')) &< \frac{\epsilon}{3}. \end{aligned}$$

  Then

$$d_X(a_N, a'_N) \le d_X(a_N, x) + d_X(x, x') + d_X(x', a'_N).$$

Hence, if $d_X\left(x, x'\right) < \delta/3$, then $d_X\left(a_N, a'_N\right) < \delta$, and

$$\begin{aligned} d_Y\left(F(x), F(x')\right) &\leq d_Y\left(F(x), f(a_N)\right) + d_Y\left(f(a_N), f(a'_N)\right) \\ &\quad + d_Y\left(f(a'_N), F(x')\right) < \epsilon. \end{aligned}$$

It follows that F is uniformly continuous on X

Problem 3. Fix a prime number p. For any nonzero rational number $r \in \mathbb{Q}$ there is a unique integer $k \in \mathbb{Z}$ such that $r = mp^k/n$, where m, n are integers that are not divisible by p. We then define $|r|_p = p^{-k}$. We define $|0|_p = 0$.

(a) Prove that $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ satisfies:

1. $|r|_p \geq 0$ and $|r|_p = 0$ if and only if $r = 0$;
2. $|-r|_p = |r_p|$;
3. $|r+s|_p \leq \max\{|r|_p, |s|_p\}$.

Deduce that $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by

$$d(r, s) = |r - s|_p$$

is an ultrametric on $\mathbb{Q}$. Show that $(\mathbb{Q}, d)$ is not complete.

(b) Let $(\mathbb{Q}_p, d_p)$ denote the completion of $(\mathbb{Q}, d)$. Use the result of Problem 2 to prove that addition $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ extends to a unique continuous function $+_p : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$.

Solution.

- (a) The first two properties follow directly from the definition. To prove the last property, suppose that $r = mp^k/n$ and $s = ap^j/b$, where m, n, a, b are integers that are not divisible by p and $j \geq k$. Then

$$r + s = \left(\frac{bm + anp^{j-k}}{bn}\right) p^k.$$

Since $d = bn$ is not divisible by p, we have

$$r + s = \frac{cp^\ell}{d}$$

where $\ell \geq k = \max\{j, k\}$, so

$$|r+s|_p = \frac{1}{p^\ell} \leq \max\left\{\frac{1}{p^k}, \frac{1}{p^j}\right\} = \max\{|r|_p, |s|_p\}.$$

- The properties of $|\cdot|_p$ imply immediately that d is an ultrametric.

- We only give a brief outline of a proof that $(\mathbb{Q}, d)$ is not complete. (Perhaps there is a simpler one.) We use a long-division algorithm to prove that every rational number $r \in \mathbb{Q}$ has a $|\cdot|_p$-convergent expansion of the form

$$r = \sum_{i=k}^{\infty} r_i p^i \qquad \text{with } k, r_i \in \mathbb{Z} \text{ and } 0 \leq r_i \leq p-1 \tag{1}$$

in which the r_i are *periodic* functions of i.[1] For example,

$$\frac{1}{1-p} = \sum_{i=0}^{\infty} p^i \in \mathbb{Q}.$$

It then follows that the partial sums of a series with non-periodic coefficients cannot converge to any rational number. For example, the series (r_n) defined by

$$r_n = \sum_{i=1}^{n} p^{i!} = p + p^2 + p^6 + \ldots + p^{n!}$$

is Cauchy in $\mathbb{Q}_p$ but does not converge to any $r \in \mathbb{Q}$.

- The completion $\mathbb{Q}_p$ may be thought of concretely as the space of all sequences of the form (1).

- (b) We equip $\mathbb{Q} \times \mathbb{Q}$ with the product metric $d_{\mathbb{Q}\times\mathbb{Q}} = d \times d$,

$$d_{\mathbb{Q}\times\mathbb{Q}}\left((r, s), (r', s')\right) = d(r, r') + d(s, s').$$

We will temporarily use the notation $+(r, s) = r + s$ for the addition function.

[1] To obtain this algorithm, write

$$m = \sum_{i=0}^{M} m_i p^i, \qquad n = \sum_{i=0}^{N} n_i p^i, \qquad r = \sum_{i \geq k} r_i p^i$$

where $0 \leq m_i, n_i, r_i \leq p-1$ and $m_0, n_0 \neq 0$, multiply the series in the equation $nr = mp^k$, carry multiples of powers of p to the succeeding terms so that all coefficients of p^i are between 0 and $p-1$, and equate coefficients of p^i. One finds that r_i is determined by the previous N coefficients $\{r_{i-1}, \ldots, r_{i-N}\}$. Since there are only finitely many such sequences, a sequence must eventually repeat, and then the coefficients r_i will repeat.

- If $(r,s),(r',s') \in \mathbb{Q}\times\mathbb{Q}$, then by the ultrametric property of $|\cdot|_p$,

$$\begin{aligned} d\left(+(r,s),+(r',s')\right) &= \left|r+s-(r'+s')\right|_p \\ &= \left|r-r'+s-s')\right|_p \\ &\leq \max\left\{\left|r-r'\right|_p, \left|s-s')\right|_p\right\} \\ &\leq \left|r-r'\right|_p + \left|s-s')\right|_p \\ &\leq d_{\mathbb{Q}\times\mathbb{Q}}\left((r,s),(r',s')\right), \end{aligned}$$

  which proves that $+ : \mathbb{Q}\times\mathbb{Q} \to \mathbb{R}$ is uniformly continuous.

- If (X, d_X) and (Y, d_Y) are metric spaces with completions $(\tilde{X}, d_{\tilde{X}})$ and $(\tilde{Y}, d_{\tilde{Y}})$, respectively, then the completion of $(X\times Y, d_X\times d_Y)$ is $(\tilde{X}\times\tilde{Y}, d_{\tilde{X}}\times d_{\tilde{Y}})$. (The proof is left as a exercise.) Thus, the completion of $(\mathbb{Q}\times\mathbb{Q}, d\times d)$ is $(\mathbb{Q}_p\times\mathbb{Q}_p, d_p\times d_p)$, and it follows from the result of Problem 2 that $+ : \mathbb{Q}\times\mathbb{Q} \to \mathbb{R}$ extends to a unique uniformly continuous map $+|_p : \mathbb{Q}_p\times\mathbb{Q}_p \to \mathbb{R}$.

- The sum of p-adic numbers $r, s \in \mathbb{Q}_p$ may be computed by adding their series expansions and carrying any multiples of p. For example, if

$$r = 1 + 0p + 0p^2 + \ldots,$$
$$s = (p-1) + (p-1)p + (p-1)p^2 + \ldots,$$

  then $r+s=0$, so s is the additive inverse of 1 in $\mathbb{Q}_p$, meaning that the sum of the series for s is -1.

Remark. Elements of $\mathbb{Q}_p$ are called p-adic numbers, important in algebraic number theory. Multiplication of rational numbers also extends continuously from $\mathbb{Q}$ to $\mathbb{Q}_p$, so $\mathbb{Q}_p$ is a complete field. Analysis on $(\mathbb{Q}_p, |\cdot|_p)$ is an ultrametric analog of the more familiar analysis on the Euclidean real line $(\mathbb{R}, |\cdot|)$.

Problem 4. A metric space is said to be: *connected* if it is not the union of two disjoint non-empty open sets; *totally disconnected* if the only non-empty connected subspaces consist of a single point; and *perfect* if every point in the space is an accumulation point, meaning that it is a limit of a sequence of other points in the space.

Let $X = \{0,1\}^{\mathbb{N}}$ be the space of all sequences consisting of zeros or ones:

$$X = \{(s_1, s_2, s_3, \ldots) \mid s_n \in \{0,1\}\}.$$

Define $d : X \times X \to \mathbb{R}$ by

$$d(\mathbf{s}, \mathbf{t}) = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}$$

where $\mathbf{s} = (s_1, s_2, s_3, \ldots)$, $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, and

$$\delta_n = \begin{cases} 0 & \text{if } s_n = t_n, \\ 1 & \text{if } s_n \neq t_n. \end{cases}$$

(a) Prove that d is a metric on X.

(b) Prove that X is compact, totally disconnected, and perfect.

(c) Prove that the Cantor set C, regarded as a metric subspace of $[0,1]$ with the standard metric, is homeomorphic to X. (You can assume that the Cantor set is in one-to-one correspondence with the set of numbers that have a base-three expansion $0.b_1b_2b_3\ldots$ with no 1's, and that for any such number this base-three expansion is unique.)

(d) Define the shift map $\sigma : X \to X$ by

$$\sigma(s_1, s_2, s_3, \ldots) = (s_2, s_3, s_4, \ldots).$$

Prove that σ is continuous.

(e) Let $\sigma^n = \sigma \circ \sigma \circ \ldots \circ \sigma$ denote the n-fold composition of σ with itself. Show that there exists a $\delta > 0$ such that for any $\mathbf{s} \in X$ and any neighborhood U of $\mathbf{s}$, there exists $\mathbf{t} \in U$ and $n \in \mathbb{N}$ with

$$d\left(\sigma^n(\mathbf{s}), \sigma^n(\mathbf{t})\right) > \delta.$$

(f) Prove that there is a point $\mathbf{s} \in X$ such that the orbit of $\mathbf{s}$ under σ,

$$\{\sigma^n(\mathbf{s}) \mid n = 0, 1, 2, \ldots\},$$

is dense in X.

Solution.

- (a) The series defining d converges, since the terms are bounded by the terms 2^{-n} of a convergent geometric series. The function d is clearly symmetric and nonnegative. If $d(\mathbf{s}, \mathbf{t}) = 0$, then $\delta_n = 0$ and $s_n = t_n$ for every $n \in \mathbb{N}$, so $\mathbf{s} = \mathbf{t}$. Finally, if $\mathbf{s} = (s_1, s_2, s_3, \ldots)$, $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, $\mathbf{r} = (r_1, r_2, r_3, \ldots)$, and

$$\delta_n = \begin{cases} 0 & \text{if } s_n = t_n, \\ 1 & \text{if } s_n \neq t_n, \end{cases} \quad \epsilon_n = \begin{cases} 0 & \text{if } t_n = r_n, \\ 1 & \text{if } t_n \neq r_n, \end{cases} \quad \eta_n = \begin{cases} 0 & \text{if } s_n = r_n, \\ 1 & \text{if } s_n \neq r_n, \end{cases}$$

  then $\eta_n \leq \delta_n + \epsilon_n$, so $d(\mathbf{s}, \mathbf{r}) \leq d(\mathbf{s}, \mathbf{t}) + d(\mathbf{t}, \mathbf{r})$.

- (b) To show that X is compact, we prove that it is complete and totally bounded. We use the following properties of the metric:

  1. if $d(\mathbf{s}, \mathbf{t}) < 1/2^N$ then $s_n = t_n$ for $1 \leq n \leq N$;
  2. if $\epsilon > 1/2^N$ and $s_n = t_n$ for $1 \leq n \leq N$ then $d(\mathbf{s}, \mathbf{t}) < \epsilon$.

- Suppose that $(\mathbf{s}_k)_{k=1}^{\infty}$ is a Cauchy sequence in X. We write

$$\mathbf{s}_k = (s_{1,k}, s_{2,k}, \ldots, s_{n,k}, \ldots).$$

  For any $N \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that $d(\mathbf{s}_j, \mathbf{s}_k) < 1/2^N$ for all $j, k > K$, and hence $s_{n,j} = s_{n,k}$ for all $n \leq N$. Thus, the terms in the sequences are eventually the same. We denote the eventual common value of $(s_{n,k})_{k=1}^{\infty}$ by s_n, and define

$$\mathbf{s} = (s_1, s_2, \ldots, s_n, \ldots).$$

  Then for $k > K$ we have

$$\begin{aligned} d(\mathbf{s}_k, \mathbf{s}) &\leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &\leq \frac{1}{2^N}. \end{aligned}$$

  It follows that $\mathbf{s}_k \to \mathbf{s}$ as $k \to \infty$, so X is complete.

- Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\epsilon > 1/2^N$. If

$$\mathbf{s} = (s_1, s_2, s_3, \dots, s_N, s_{N+1}, s_{N+2}, \dots)$$

then the open ball $B_\epsilon(\mathbf{s})$ contains all sequences of the form

$$\mathbf{t} = (s_1, \dots, s_N, t_{N+1}, t_{N+2} \dots) \qquad \text{where } t_n \in \{0, 1\} \text{ for } n > N.$$

There are 2^N initial sequences $(s_1, \dots, s_N) \in \{0, 1\}^N$, so X is covered by 2^N balls of radius ϵ, and X is totally bounded.

- Next we show that X is totally disconnected. Given $N \in \mathbb{N}$, let

$$U_N = \{\mathbf{s} \in X \mid s_N = 0\}, \qquad V_N = \{\mathbf{s} \in X \mid s_N = 1\}.$$

Then U_N and V_N are open. For example, suppose $\mathbf{s} \in U_N$. If $d(\mathbf{s}, \mathbf{t}) < 1/2^N$ then $t_N = s_N = 0$, so $\mathbf{t} \in U_N$, meaning that $B_{1/2^N}(\mathbf{s}) \subset U_N$, so U_N is open. (It follows that $X = U_N \cup V_N$ is the disjoint union of non-empty open sets so it is not connected.)

- Suppose that $A \subset X$ contains at least two distinct points $\mathbf{s}$, $\mathbf{t}$. There exists $N \in \mathbb{N}$ such that $s_N \neq t_N$. Let $U = A \cap U_N$ and $V = A \cap V_N$. Then U, V are disjoint, non-empty sets that are open in A, and $A = U \cup V$. Hence A is not connected and X is totally disconnected.

- Suppose that $\mathbf{s} = (s_1, s_2, \dots, s_n, \dots) \in X$ and define

$$\mathbf{s}_k = (s_{1,k}, s_{2,k}, \dots, s_{n,k}, \dots) \in X$$

by $s_{n,k} = s_n$ if $n \neq k$ and $s_{n,k} \neq s_n$ if $n = k$. Thus, $\mathbf{s}_k$ differs from $\mathbf{s}$ only in the kth term. Then $\mathbf{s}_k \neq \mathbf{s}$ and $d(\mathbf{s}_k, \mathbf{s}) = 1/2^k$. Hence $\mathbf{s}_k \to \mathbf{s}$ as $k \to \infty$, and X is perfect.

- Compare the Cantor set with the closed interval $[0, 1]$ — a compact space that is perfect but connected — and the finite set $\{1, 2, \dots, N\}$ with the discrete metric — a compact space that is totally disconnected but not perfect.

- (c) If $a \in C$ has the base-three expansion $a = 0.a_1a_2a_3\dots$ with no 1's, we define $\phi(a) = \mathbf{s} \in X$ where $\mathbf{s} = (s_1, s_2, s_3, \dots)$ is given by

$$s_n = \begin{cases} 0 & \text{if } a_n = 0, \\ 1 & \text{if } a_n = 2. \end{cases}$$

Then $\phi : C \to X$ is one-to-one and onto. We need to show that ϕ and its inverse are continuous.

- The metric on $C \subset [0,1]$ is the Euclidean metric $d(a,b) = |a-b|$. Suppose that $a = 0.a_1a_2a_3\ldots$, $b = 0.b_1b_2b_3\ldots$ are the base-three expansions (without 1's) of $a, b \in C$. If $a_n = b_n$ for $1 \le n \le N$, then

$$|a-b| \le \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \frac{1}{3^N},$$

since the furthest such numbers (the endpoints of one of the closed intervals in the construction of the Cantor set) have the expansions $0.a_1\ldots a_N222\ldots$, $0.a_1\ldots a_N000\ldots$. If $a_N \ne b_N$, then

$$|a-b| \ge \frac{2}{3^N} - \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \frac{1}{3^N},$$

since the closest such numbers have the expansions $0.a_1, \ldots, a_{N-1}2000\ldots$, $0.a_1, \ldots, a_{N-1}0222\ldots$.

- Let $\phi(a) = \mathbf{s}$, $\phi(b) = \mathbf{t}$. If $N \in \mathbb{N}$ and $|a-b| < 1/3^N$, then $a_n = b_n$ for $1 \le n \le N$, so $s_n = t_n$ for $1 \le n \le N$, and $d(\phi(a), \phi(b)) \le 1/2^N$. This proves that $\phi : C \to X$ is continuous.

- If $d(\mathbf{s}, \mathbf{t}) < 1/2^N$, then $s_n = t_n$ for $1 \le n \le N$, so $a_n = b_n$ for $1 \le n \le N$, and $|\phi^{-1}(\mathbf{s}) - \phi^{-1}(\mathbf{t})| \le 1/3^N$. This proves that $\phi^{-1} : X \to C$ is continuous. Thus, ϕ is a homeomorphism.

- (d) We have

$$d\left(\sigma(\mathbf{s}), \sigma(\mathbf{t})\right) = \sum_{n=1}^{\infty} \frac{\delta_{n+1}}{2^n} = 2\sum_{n=2}^{\infty} \frac{\delta_n}{2^n} \le 2d\left(\mathbf{s}, \mathbf{t}\right),$$

which implies that σ is continuous.

- (e) The stated condition holds for any $0 < \delta < 1$. If U is a neighborhood of $\mathbf{s} \in X$, then U contains a ball $B_\epsilon(\mathbf{s})$ for some $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/2^N < \epsilon$, and define $\mathbf{t} \in X$ by $t_n = s_n$ for $1 \le n \le N$ and

$t_n \neq s_n$ for $n \geq N+1$. Then $d(\mathbf{s}, \mathbf{t}) = 1/2^N < \epsilon$, so $\mathbf{t} \in U$. On the other hand,

$$\sigma^N(\mathbf{s}) = (s_{N+1}, s_{N+2}, \dots), \qquad \sigma^N(\mathbf{t}) = (t_{N+1}, t_{N+2}, \dots)$$

differ in every term, so

$$d\left(\sigma^N(\mathbf{s}), \sigma^N(\mathbf{s}\right) = 1 > \delta.$$

- (f) Define $\mathbf{s} \in X$ by listing all one-term sequences, followed by all two-term sequences, followed by all three-term sequences, and so on. For example, we could define

  $$\mathbf{s} = (0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, \dots).$$

  Then for every $\mathbf{t} \in X$ and every $N \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $\sigma^n(\mathbf{s})$ and $\mathbf{t}$ have the same first N terms. Hence, $d(\sigma^n(\mathbf{s}), \mathbf{t}) \leq 1/2^N$, which proves that the orbit of $\mathbf{s}$ under σ is dense in X.

Remark. The metric space X is an example of a symbol space. Such symbol spaces arise in the study of chaotic dynamical systems, and the representation of chaotic dynamical systems by shift maps on symbol spaces is called symbolic dynamics. In this context, the property in (e) is called 'sensitive dependence on initial conditions.'

For example, consider the logistic map $F_\mu : \Lambda \subset [0,1] \to [0,1]$ defined by $F_\mu(x) = \mu x(1-x)$. For $\mu > 4$ the map F_μ has an invariant Cantor set Λ of points that remain in $[0,1]$ under all iterates F_μ^n. One can prove that there is a homeomorphism $\phi : \Lambda \to X$ such that $F_\mu = \phi^{-1} \circ \sigma \circ \phi$ is topologically conjugate to the shift map σ.