Problem 1. Give an example of a sequence \((f_n)\) of continuous functions \(f_n : [0, 1] \to \mathbb{R}\) that converges to 0 with respect to the \(L^1\)-norm,

\[
\|f\|_1 = \int_0^1 |f(x)| \, dx,
\]
such that the real sequence of pointwise values \((f_n(x))\) does not converge for any \(0 \leq x \leq 1\). Verify that there is a subsequence that converges pointwise-a.e. to 0.

Solution.

• Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function such that

\[
f(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1, \\
0 & \text{if } x \leq -1 \text{ or } x \geq 2.
\end{cases}
\]

Such a function exists by Urysohn’s lemma; an explicit example is the piecewise-linear function

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq -1, \\
x + 1 & \text{if } -1 < x < 0, \\
1 & \text{if } 0 \leq x \leq 1, \\
2 - x & \text{if } 1 < x < 2, \\
0 & \text{if } x \geq 2.
\end{cases}
\]

• We may write \(n \in \mathbb{N}\) uniquely as \(n = 2^m + k\) where \(m = 0, 1, 2, \ldots\) and \(k = 0, 1, \ldots, 2^m - 1\). We then define \(f_n : [0, 1] \to \mathbb{R}\) by

\[
f_n(x) = f(2^m x - k).
\]

The graphs of the functions in the sequence \((f_n)\) consist of thinner and thinner ‘plateaus’ of height 1 that sweep successively across the interval [0, 1].
• Making the change of variables \( t = 2^m x - k \), we compute that

\[
\int_0^1 |f_n(x)| \, dx \leq \frac{1}{2^m} \int_{\mathbb{R}} |f(t)| \, dt \to 0 \quad \text{as } n \to \infty.
\]

Hence, \( f_n \to 0 \) with respect to \( \| \cdot \|_1 \).

• For any \( x \in [0,1] \), the sequence \( (f_n(x)) \) of pointwise values contains infinitely many 0’s and infinitely many 1’s, so it does not converge.

• The subsequence \( (g_m) \) of \( (f_n) \) defined by \( g_m(x) = f(2^m x) \) converges pointwise to 0 except at \( x = 0 \), where it converges to 1. Since a set consisting of a single point has Lebesgue measure zero, \( (g_m) \) converges pointwise a.e. to 0 on \( [0,1] \).

**Problem 2.** The sequence space \( \ell^\infty \) is the Banach space of all bounded real sequences,

\[
\ell^\infty = \{(x_1, x_2, \ldots, x_n, \ldots) \mid x_n \in \mathbb{R}, \exists M \in \mathbb{R} \text{ s.t. } |x_n| \leq M \text{ for all } n \in \mathbb{N}\},
\]

with the norm

\[
\|(x_1, x_2, \ldots, x_n, \ldots)\| = \sup_{n \in \mathbb{N}} |x_n|.
\]

Let

\[
B = \{(x_1, x_2, \ldots, x_n, \ldots) \mid 0 \leq x_n \leq 1 \text{ for all } n \in \mathbb{N}\}.
\]

Show that \( B \) is a closed, bounded subset of \( \ell^\infty \) that is not compact. (You don’t need to verify that \( \ell^\infty \) is a Banach space.)

**Solution.**

• We have \( \|x\| \leq 1 \) for every \( x \in B \), so \( B \) is bounded.

• Suppose that \( (x_k)_{k=1}^{\infty} \) is a convergent sequence in \( B \), with limit \( y \in \ell^\infty \). We write \( x_k = (x_{n,k})_{n=1}^{\infty} \) and \( y = (y_n)_{n=1}^{\infty} \). The definition of the norm implies that

\[
|x_{n,k} - y_n| \leq \|x_k - y\|
\]

for every \( n \in \mathbb{N} \), so \( x_{n,k} \to y_n \) in \( \mathbb{R} \) as \( k \to \infty \). Since \( x_k \in B \), we have \( 0 \leq x_{n,k} \leq 1 \), and taking the limit of this equation as \( k \to \infty \) we get \( 0 \leq y_n \leq 1 \). It follows that \( y \in B \), so \( B \) is closed.
• Define a sequence \((x_k)_{k=1}^\infty\) in \(B\) by \(x_k = (x_{n,k})_{n=1}^\infty\) with

\[
x_{n,k} = \begin{cases} 
1 & \text{if } n = k, \\
0 & \text{if } n \neq k.
\end{cases}
\]

That is, \(x_1 = (1, 0, 0, \ldots)\), \(x_2 = (0, 1, 0, \ldots)\), \(x_3 = (0, 0, 1, \ldots)\), and so on. Then \(\|x_j - x_k\| = 1\) for every \(j \neq k\). It follows that no subsequence of \((x_k)\) is Cauchy, so no subsequence converges. Therefore \(B\) is not compact.

**Problem 3.** Suppose that \((x_n)\) is a sequence in a compact metric space with the property that every convergent subsequence has the same limit \(x\). Prove that \(x_n \to x\) as \(n \to \infty\).

**Solution.**

• Suppose for contradiction that the sequence does not converge to \(x\). Then there exists a neighborhood \(U\) of \(x\) such that infinitely many terms of the sequence do not belong to \(U\). We may therefore pick a subsequence \((y_k)\), with \(y_k = x_{n_k}\), such that \(y_k \notin U\) for any \(k \in \mathbb{N}\).

• Since the metric space is compact, the sequence \((y_k)\) has a convergent subsequence \((z_j)\), with \(z_j = y_{k_j}\). This sequence \((z_j)\) is a convergent subsequence of the original sequence \((x_n)\), but it cannot converge to \(x\) since \(z_j \notin U\) for any \(j\). This contradiction proves the result.

**Problem 4.** (a) Prove that a real-valued function \(f : X \to \mathbb{R}\) is sequentially lower semi-continuous on \(X\) if and only if for every \(a \in \mathbb{R}\) the set \(f^{-1}((a, \infty))\) is open in \(X\).

(b) If \(X\) is a compact metric space and \(f : X \to \mathbb{R}\) is sequentially lower semi-continuous on \(X\), prove that \(f\) is bounded from below and attains its minimum value. Give examples to show that such an \(f\) need not be bounded from above and need not attain its supremum even if it is.

**Solution.**
(a) Suppose that $f^{-1}((a, \infty))$ is open in $X$ for every $a \in \mathbb{R}$. Let $(x_n)$ be a sequence in $X$ converging to $x$. If

$$f(x) > a,$$

then $U = f^{-1}((a, \infty))$ is an open neighborhood of $x$, so $x_n \in U$ for all sufficiently large $n$. Hence, $f(x_n) > a$ for all sufficiently large $n$, and

$$\liminf_{n \to \infty} f(x_n) \geq a.$$

It follows that

$$f(x) \leq \liminf_{n \to \infty} f(x_n),$$

(otherwise choosing $f(x) > a > \liminf_{n \to \infty} f(x_n)$ leads to a contradiction), so $f$ is sequentially lower semi-continuous.

(b) The collection of intervals $\{(a, \infty) \mid a \in \mathbb{R}\}$ covers $\mathbb{R}$, so

$$\{f^{-1}((a, \infty)) \mid a \in \mathbb{R}\}$$

covers $X$. Since $f$ is lower semi-continuous this is an open cover of $X$, and since $X$ is compact there is a finite subcover

$$\{f^{-1}((a_n, \infty)) \mid 1 \leq n \leq N\}.$$

It follows that

$$f(X) \subset \bigcup_{n=1}^{N} (a_n, \infty),$$

so $f$ is bounded from below by $\min \{a_n \mid 1 \leq n \leq N\}$. 

• For the converse, suppose that $f^{-1}((a, \infty))$ is not open in $X$ for some $a \in \mathbb{R}$. Then the complement

$$f^{-1}((a, \infty))^c = f^{-1}((\infty, a])$$

is not closed, and there exists a sequence $(x_n)$ in $f^{-1}((\infty, a])$ converging to $x \in f^{-1}((a, \infty))$. Since $f(x_n) \leq a$ and $f(x) > a$,

$$f(x) > \liminf_{n \to \infty} f(x_n),$$

so $f$ is not sequentially lower semi-continuous.
Let
\[ m = \inf_{x \in X} f(x). \]  
(1)
This infimum is finite since \( f \) is bounded from below. Choose a sequence \( (x_n)_{n=1}^{\infty} \) in \( X \) such that \( f(x_n) \to m \). Since \( X \) is compact there exists a convergent subsequence \( (x_{n_k})_{k=1}^{\infty} \). Let
\[ x = \lim_{k \to \infty} x_{n_k}. \]

By the sequential lower semi-continuity of \( f \), we have
\[ f(x) \leq \liminf_{k \to \infty} f(x_{n_k}) \leq m. \]
On the other hand, by the definition of \( m \) in (1), we have
\[ f(x) \geq m. \]
It follows that \( f(x) = m \), so \( f \) attains its infimum on \( X \).

Define functions \( f, g : [0, 1] \to \mathbb{R} \) by
\[
 f(x) = \begin{cases} 
 1/x & \text{if } 0 < x \leq 1, \\
 0 & \text{if } x = 0,
\end{cases} \\
 g(x) = \begin{cases} 
 x & \text{if } 0 \leq x < 1, \\
 0 & \text{if } x = 1.
\end{cases}
\]
Then \( f, g \) are lower semi-continuous on the compact set \([0, 1]\). The function \( f \) is not bounded from above, and \( g \) is bounded from above with
\[ \sup_{x \in [0,1]} g(x) = 1, \]
but \( g(x) \neq 1 \) for any \( x \in [0,1] \).