Solutions: Problem Set 4
Math 201A, Fall 2006

Problem 1. Give an example of a sequence (f_n) of continuous functions $f_n : [0,1] \to \mathbb{R}$ that converges to 0 with respect to the L^1-norm,

$$\|f\|_1 = \int_0^1 |f(x)|\,dx,$$

such that the real sequence of pointwise values $(f_n(x))$ does not converge for any $0 \le x \le 1$. Verify that there is a subsequence that converges pointwise-a.e. to 0.

Solution.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

  $$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x \le -1 \text{ or } x \ge 2. \end{cases}$$

  Such a function exists by Urysohn's lemma; an explicit example is the piecewise-linear function

  $$f(x) = \begin{cases} 0 & \text{if } x \le -1, \\ x+1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 \le x \le 1, \\ 2-x & \text{if } 1 < x < 2, \\ 0 & \text{if } x \ge 2. \end{cases}$$

- We may write $n \in \mathbb{N}$ uniquely as $n = 2^m + k$ where $m = 0, 1, 2, \dots$ and $k = 0, 1, \dots, 2^m - 1$. We then define $f_n : [0,1] \to \mathbb{R}$ by

  $$f_n(x) = f\left(2^m x - k\right).$$

  The graphs of the functions in the sequence (f_n) consist of thinner and thinner 'plateaus' of height 1 that sweep successively across the interval $[0,1]$.

- Making the change of variables $t = 2^m x - k$, we compute that

$$\int_0^1 |f_n(x)| \, dx \leq \frac{1}{2^m} \int_{\mathbb{R}} |f(t)| \, dt \to 0 \qquad \text{as } n \to \infty.$$

  Hence, $f_n \to 0$ with respect to $\|\cdot\|_1$.

- For any $x \in [0,1]$, the sequence $(f_n(x))$ of pointwise values contains infinitely many 0's and infinitely many 1's, so it does not converge.

- The subsequence (g_m) of (f_n) defined by $g_m(x) = f(2^m x)$ converges pointwise to 0 except at $x = 0$, where it converges to 1. Since a set consisting of a single point has Lebesgue measure zero, (g_m) converges pointwise a.e. to 0 on $[0,1]$.

Problem 2. The sequence space ℓ^∞ is the Banach space of all bounded real sequences,

$$\ell^\infty = \{(x_1, x_2, \ldots, x_n, \ldots) \mid x_n \in \mathbb{R}, \exists M \in \mathbb{R} \text{ s.t. } |x_n| \leq M \text{ for all } n \in \mathbb{N}\},$$

with the norm

$$\|(x_1, x_2, \ldots, x_n, \ldots)\| = \sup_{n \in \mathbb{N}} |x_n|.$$

Let

$$B = \{(x_1, x_2, \ldots, x_n, \ldots) \mid 0 \leq x_n \leq 1 \text{ for all } n \in \mathbb{N}\}.$$

Show that B is a closed, bounded subset of ℓ^∞ that is not compact. (You don't need to verify that ℓ^∞ is a Banach space.)

Solution.

- We have $\|x\| \leq 1$ for every $x \in B$, so B is bounded.

- Suppose that $(x_k)_{k=1}^\infty$ is a convergent sequence in B, with limit $y \in \ell^\infty$. We write $x_k = (x_{n,k})_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$. The definition of the norm implies that

$$|x_{n,k} - y_n| \leq \|x_k - y\|,$$

  for every $n \in \mathbb{N}$, so $x_{n,k} \to y_n$ in $\mathbb{R}$ as $k \to \infty$. Since $x_k \in B$, we have $0 \leq x_{n,k} \leq 1$, and taking the limit of this equation as $k \to \infty$ we get $0 \leq y_n \leq 1$. It follows that $y \in B$, so B is closed.

- Define a sequence $(x_k)_{k=1}^{\infty}$ in B by $x_k = (x_{n,k})_{n=1}^{\infty}$ with

$$x_{n,k} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

  That is, $x_1 = (1, 0, 0, \ldots)$, $x_2 = (0, 1, 0, \ldots)$, $x_3 = (0, 0, 1, \ldots)$, and so on. Then $\|x_j - x_k\| = 1$ for every $j \neq k$. It follows that no subsequence of (x_k) is Cauchy, so no subsequence converges. Therefore B is not compact.

Problem 3. Suppose that (x_n) is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.

Solution.

- Suppose for contradiction that the sequence does not converge to x. Then there exists a neighborhood U of x such that infinitely many terms of the sequence do not belong to U. We may therefore pick a subsequence (y_k), with $y_k = x_{n_k}$, such that $y_k \notin U$ for any $k \in \mathbb{N}$.

- Since the metric space is compact, the sequence (y_k) has a convergent subsequence (z_j), with $z_j = y_{k_j}$. This sequence (z_j) is a convergent subsequence of the original sequence (x_n), but it cannot converge to x since $z_j \notin U$ for any j. This contradiction proves the result.

Problem 4. (a) Prove that a real-valued function $f : X \to \mathbb{R}$ is sequentially lower semi-continuous on X if and only if for every $a \in \mathbb{R}$ the set $f^{-1}((a, \infty))$ is open in X.
(b) If X is a compact metric space and $f : X \to \mathbb{R}$ is sequentially lower semi-continuous on X, prove that f is bounded from below and attains its minimum value. Give examples to show that such an f need not be bounded from above and need not attain its supremum even if it is.

Solution.

- (a) Suppose that $f^{-1}((a,\infty))$ is open in X for every $a \in \mathbb{R}$. Let (x_n) be a sequence in X converging to x. If

$$f(x) > a,$$

then $U = f^{-1}((a,\infty))$ is an open neighborhood of x, so $x_n \in U$ for all sufficiently large n. Hence, $f(x_n) > a$ for all sufficiently large n, and

$$\liminf_{n\to\infty} f(x_n) \geq a.$$

It follows that

$$f(x) \leq \liminf_{n\to\infty} f(x_n),$$

(otherwise choosing $f(x) > a > \liminf_{n\to\infty} f(x_n)$ leads to a contradiction), so f is sequentially lower semi-continuous.

- For the converse, suppose that $f^{-1}((a,\infty))$ is not open in X for some $a \in \mathbb{R}$. Then the complement

$$f^{-1}((a,\infty))^c = f^{-1}((-\infty,a])$$

is not closed, and there exists a sequence (x_n) in $f^{-1}((-\infty,a])$ converging to $x \in f^{-1}((a,\infty))$. Since $f(x_n) \leq a$ and $f(x) > a$,

$$f(x) > \liminf_{n\to\infty} f(x_n),$$

so f is not sequentially lower semi-continuous.

- (b) The collection of intervals $\{(a,\infty) \mid a \in \mathbb{R}\}$ covers $\mathbb{R}$, so

$$\left\{f^{-1}((a,\infty)) \mid a \in \mathbb{R}\right\}$$

covers X. Since f is lower semi-continuous this is an open cover of X, and since X is compact there is a finite subcover

$$\left\{f^{-1}((a_n,\infty)) \mid 1 \leq n \leq N\right\}.$$

It follows that

$$f(X) \subset \bigcup_{n=1}^{N} (a_n,\infty),$$

so f is bounded from below by $\min\{a_n \mid 1 \leq n \leq N\}$.

- Let
$$m = \inf_{x \in X} f(x). \tag{1}$$
This infimum is finite since f is bounded from below. Choose a sequence $(x_n)_{n=1}^{\infty}$ in X such that $f(x_n) \to m$. Since X is compact there exists a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Let
$$x = \lim_{k \to \infty} x_{n_k}.$$

- By the sequential lower semi-continuity of f, we have
$$\begin{aligned} f(x) &\leq \liminf_{k \to \infty} f(x_{n_k}) \\ &\leq m. \end{aligned}$$
On the other hand, by the definition of m in (1), we have
$$f(x) \geq m.$$
It follows that $f(x) = m$, so f attains its infimum on X.

- Define functions $f, g : [0,1] \to \mathbb{R}$ by
$$\begin{aligned} f(x) &= \begin{cases} 1/x & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases} \\ g(x) &= \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases} \end{aligned}$$
Then f, g are lower semi-continuous on the compact set $[0,1]$. The function f is not bounded from above, and g is bounded from above with
$$\sup_{x \in [0,1]} g(x) = 1,$$
but $g(x) \neq 1$ for any $x \in [0,1]$.