Solutions: Problem Set 5
Math 201A, Fall 2006

1. Suppose that (x_n) is a bounded sequence of real numbers. Define a sequence (y_n) by

$$y_n = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

(a) Prove that

$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} x_n.$$

(b) If (x_n) converges, must (y_n) converge? If (y_n) converges, must (x_n) converge? Prove your answers.

Solution.

- (a) It suffices to prove that $\limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} x_n$, since an application of this inequality to $(-x_n)$ implies that $\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n$, and we always have $\liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n$.

- Let $L = \limsup_{n\to\infty} x_n$. If $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n < L + \epsilon$ for all $n > N$. It follows that for $n > N$

$$\begin{aligned} y_n &= \frac{x_1 + x_2 + \ldots + x_N}{n} + \frac{x_{N+1} + x_2 + \ldots + x_n}{n} \\ &\le \frac{x_1 + x_2 + \ldots + x_N}{n} + L + \epsilon. \end{aligned}$$

  Hence,

$$\sup_{k\ge n} y_k \le \frac{x_1 + x_2 + \ldots + x_N}{n} + L + \epsilon.$$

  Taking the limit as $n \to \infty$, we get

$$\limsup_{n\to\infty} y_n \le L + \epsilon.$$

  Since this holds for arbitrary $\epsilon > 0$, we conclude that

$$\limsup_{n\to\infty} y_n \le L,$$

  which proves the result.

- (b) If (x_n) converges, then $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, and (a) implies that $\liminf_{n\to\infty} y_n = \limsup_{n\to\infty} y_n$, so (y_n) converges to the same limit as (x_n).

- The convergence of (y_n) does not imply the convergence of (x_n). For example, if $x_n = (-1)^{n+1}$, then

  $$y_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

  Thus, (x_n) does not converge, but (y_n) converges to 0.

2. Let A be a subset of a metric space X. Define the characteristic function $\chi_A : X \to \mathbb{R}$ of A by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that χ_A is lower semi-continuous if and only if A is open.

Solution.

- First suppose that A is open. If $x \in A$, then $\chi_A(x) = 1$ and there is a neighborhood of x contained in A. Hence, if $x_n \to x$, then there exists N such that $x_n \in A$ for all $n \geq N$. It follows that $\chi_A(x_n) = 1$ for all $n \geq N$ and

  $$\chi_A(x) = \liminf_{n\to\infty} \chi_A(x_n).$$

  If $x \notin A$, then $0 = \chi_A(x) \leq \chi_A(y)$ for every $y \in X$. Hence

  $$\chi_A(x) \leq \liminf_{n\to\infty} \chi_A(x_n)$$

  for any sequence (x_n). Hence χ_A is lower semi-continuous on X.

- For the converse, suppose that A is not open. Then the complement A^c is not closed and there is a sequence (x_n) in A^c that converges to $x \in A$. It follows that $\chi_A(x_n) = 0$ for every n and $\chi_A(x) = 1$, so

  $$\liminf_{n\to\infty} \chi_A(x_n) < \chi_A(x).$$

  Hence χ_A is not lower semi-continuous.

3. Let $C_c(\mathbb{R})$ be the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. We define the sup-norm $\|\cdot\|_\infty$ and the L^1-norm $\|\cdot\|_1$ on $C_c(\mathbb{R})$ by

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|, \qquad \|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

(a) Show that $\|f\|_\infty$ and $\|f\|_1$ are finite for any $f \in C_c(\mathbb{R})$.
(b) Is $C_c(\mathbb{R})$ equipped with the sup-norm a Banach space?
(c) Let (f_n) be a sequence in $C_c(\mathbb{R})$. Answer the following questions, and give a proof or a counterexample.

1. If $f_n \to f \in C_c(\mathbb{R})$ as $n \to \infty$ with respect to the L^1-norm, does $f_n \to f$ as $n \to \infty$ with respect to the sup-norm?

2. If $f_n \to f \in C_c(\mathbb{R})$ as $n \to \infty$ with respect to the sup-norm, does $f_n \to f$ as $n \to \infty$ with respect to the L^1-norm?

Solution.

- (a) A function $f \in C_c(\mathbb{R})$ is zero outside a compact interval $[-R, R]$. Hence

  $$\|f\|_\infty = \sup_{x\in[-R,R]} |f(x)| < \infty$$

  since a continuous function on a compact set is bounded. Also, we have

  $$\|f\|_1 = \int_{-R}^{R} |f(x)|\, dx \leq 2R\|f\|_\infty < \infty.$$

- Note that the constant in this bound depends on the support of f. It is not true that there exists an $M > 0$ such that $\|f\|_1 \leq M\|f\|_\infty$ for all $f \in C_c(\mathbb{R})$.

- (b) The space $(C_c(\mathbb{R}), \|\cdot\|_\infty)$ is not complete, so it is not a Banach space.

- To give a non-convergent Cauchy sequence, let $\chi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$, and $\chi(x) = 0$ for $|x| \geq 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly positive continuous function such that $f(x) \to 0$ as $|x| \to \infty$; for example, $f(x) = 1/(1 + x^2)$. We define a sequence (f_n) in $C_c(\mathbb{R})$ by

  $$f_n(x) = \chi\left(\frac{x}{n}\right) f(x).$$

- Then $0 \le f_n(x) \le f(x)$ and

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| \le n, \\ 0 & \text{if } |x| \ge 2n. \end{cases}$$

For $m > n$ we have

$$\|f_m - f_n\|_\infty \le \sup_{|x|\ge n} f(x).$$

Since $\sup_{|x|\ge n} f(x) \to 0$ as $n \to \infty$, the sequence (f_n) is Cauchy.

- If $\|f_n - g\|_\infty \to 0$ as $n \to \infty$, then $f_n(x) \to g(x)$ for each $x \in \mathbb{R}$, so $g = f$. Since f does not have compact support, (f_n) does not converge in $C_c(\mathbb{R})$.

- (c) Both statements are false.

   1. For each $n \in \mathbb{N}$, define $f_n \in C_c(\mathbb{R})$ by

      $$f_n(x) = \begin{cases} 1 - n|x| & \text{if } |x| \le 1/n \\ 0 & \text{if } |x| > 1/n \end{cases}.$$

      Then

      $$\|f_n\|_1 = \frac{1}{n}, \qquad \|f_n\|_\infty = 1.$$

      Thus, $f_n \to 0$ with respect to the L^1-norm, but $f_n \not\to 0$ with respect to the sup-norm.

   2. For each $n \in \mathbb{N}$, define $f_n \in C_c(\mathbb{R})$ by

      $$f_n(x) = \begin{cases} 1/n & \text{if } |x| \le n \\ (n + 1 - |x|)/n & \text{if } n \le |x| \le n+1 \\ 0 & \text{if } |x| > n+1 \end{cases}.$$

      Then

      $$\|f_n\|_1 > 2n \cdot \frac{1}{n} = 2, \qquad \|f_n\|_\infty = \frac{1}{n}.$$

      Thus, $f_n \to 0$ with respect to the sup-norm, but $f_n \not\to 0$ with respect to the L^1-norm.

Remark. The completion of $C_c(\mathbb{R})$ with respect to the sup-norm is the space $C_0(\mathbb{R})$ of continuous functions that vanish at infinity. The completion of $C_c(\mathbb{R})$ with respect to the L^1-norm is the space $L^1(\mathbb{R})$ of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|\, dx < \infty$, where we identify functions that are equal almost everywhere.

4. A collection of sets has the *finite intersection property* if every finite subcollection has nonempty intersection.

(a) Prove that a metric space X is compact if and only if every collection of closed sets with the finite intersection property has non-empty intersection.

(b) Give an example of a collection of closed subsets of $(0, 1]$ (with its usual metric topology as a subset of $\mathbb{R}$) that has the finite intersection property but whose intersection is empty.

Solution.

- (a) Let $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be any collection of closed sets in X. Then
$$\bigcap_{\alpha \in \mathcal{A}} F_\alpha = \emptyset$$
if and only if
$$\bigcup_{\alpha \in \mathcal{A}} F_\alpha^c = X,$$
meaning that $\{F_\alpha^c \mid \alpha \in \mathcal{A}\}$ is an open cover of X. This cover has a finite subcover $\{F_{\alpha_n}^c \mid 1 \le n \le N\}$ if and only if the intersection of $\{F_{\alpha_n} \mid 1 \le n \le N\}$ is empty. Thus every open cover of X has a finite subcover if and only if every collection of closed sets with empty intersection has a finite subcollection with empty intersection. It follows that X is compact if and only if every collection of closed sets with the finite intersection property has non-empty intersection.

- (b) The set $F_n = (0, 1/n]$ is closed in $(0, 1]$ for each $n \in \mathbb{N}$. It has the finite intersection property since
$$F_{n_1} \cap F_{n_2} \cap \ldots \cap F_{n_r} = F_N$$
where $N = \max\{n_1, \ldots, n_r\}$. However,
$$\bigcap_{n=1}^{\infty} F_n = \emptyset.$$

5. Let ℓ^∞ be the space of real, bounded sequences,

$$\ell^\infty = \{(x_1, x_2, x_3, \ldots) \mid x_n \in \mathbb{R}, \exists M > 0 \text{ s.t. } |x_n| \leq M \text{ for all } n \in \mathbb{N}\},$$

equipped with the sup-norm

$$\|(x_1, x_2, x_3, \ldots)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Prove that the 'Hilbert cube'

$$C = \{(x_1, x_2, x_3, \ldots) \mid 0 \leq x_n \leq 1/n \text{ for every } n \in \mathbb{N}\}$$

is a compact subset of ℓ^∞. (You can assume that ℓ^∞ is a Banach space.)

Solution.

- We will prove that C is complete and totally bounded, hence compact.
- *Complete.* Since ℓ^∞ is complete, and a closed subset of a complete space is complete, it is enough to show that C is closed. Suppose that (x^k) is a sequence in C converging to $x \in \ell^\infty$ as $k \to \infty$. Let

  $$x^k = \left(x_1^k, x_2^k, \ldots, x_n^k, \ldots\right), \quad x = (x_1, x_2, \ldots, x_n, \ldots)$$

  where $x_n^k, x_n \in \mathbb{R}$. Then since $|x_n^k - x_n| \leq \|x^k - x\|_\infty$ for each $n \in \mathbb{N}$, we have $x_n^k \to x_n$ as $k \to \infty$; and since $0 \leq x_n^k \leq 1/n$ for all k, it follows that $0 \leq x_n \leq 1/n$. Hence, $x \in C$.
- *Totally bounded.* Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. The set

  $$C_N = \{(x_1, x_2, \ldots, x_N) \mid 0 \leq x_n \leq 1/n \text{ for } 1 \leq n \leq N\}$$

  is a bounded subset of the finite-dimensional space $\mathbb{R}^N$, equipped with the maximum-norm, and hence it it totally bounded. Let

  $$\left\{x^{k,N} \in \mathbb{R}^N \mid 1 \leq k \leq K\right\}$$

  be a finite ϵ-net for C_N. We write

  $$x^{k,N} = \left(x_1^{k,N}, x_2^{k,N}, \ldots, x_N^{k,N}\right),$$

and define

$$x^k = \left(x_1^{k,N}, x_2^{k,N}, \dots, x_N^{k,N}, 0, 0, \dots\right) \in \ell^\infty.$$

We claim that

$$\left\{x^k \mid 1 \le k \le K\right\}$$

is a finite ϵ-net for C. To prove this claim, let $x = (x_1, x_2, \dots, x_n, \dots)$ be any point in C. Then $x^N = (x_1, x_2, \dots, x_N) \in C_N$. Since $\left\{x^{k,N}\right\}$ is an ϵ-net for C_N, there is a $1 \le k \le K$ such that

$$\max_{1\le n\le N} \left|x_n^{k,N} - x_n\right| < \epsilon.$$

Moreover, since $x \in C$ we have

$$\left|x_n^k - x_n\right| = |x_n| \le \frac{1}{n} < \frac{1}{N} \qquad \text{for } n > N.$$

Since $1/N < \epsilon$, it follows that

$$\left\|x^k - x\right\| = \sup_{n\in\mathbb{N}} \left|x_n^k - x_n\right| < \epsilon.$$

Thus the finite collection of open balls $\{B_\epsilon(x^k)\}$ covers C, and C is totally bounded.

Remark. Contrast C with the bounded, but non-compact, unit cube

$$B = \{(x_1, x_2, x_3, \dots) \mid 0 \le x_n \le 1 \text{ for every } n \in \mathbb{N}\}.$$

This example of an infinite dimensional cube C whose sides get thinner as the dimension gets larger illustrates the heuristic idea that compact sets in a normed linear space are bounded sets that are ‘almost’ finite-dimensional.