Analysis
Math 201A, Fall 2006
Solutions: Problem Set 6

1. Let $C^1([0,1])$ denote the space of continuously differentiable functions $f : [0,1] \to \mathbb{R}$, and define

$$\|f\| = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| .$$

(a) Show that $\|\cdot\|$ is a norm on $C^1([0,1])$.
(b) Prove that $C^1([0,1])$ is a Banach space with respect to $\|\cdot\|$.

Solution.

- (a) It is easy to check that $\|\cdot\|$ is a norm. For example, denoting the sup-norm by $\|\cdot\|_\infty$, we have

$$\begin{aligned} \|f+g\| &= \|f+g\|_\infty + \|f'+g'\|_\infty \\ &\leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty \\ &\leq \|f\| + \|g\|. \end{aligned}$$

- (b) Suppose that (f_n) is a Cauchy sequence in $C^1([0,1])$ with respect to $\|\cdot\|$. Then (f_n), (f_n') are Cauchy sequences of continuous functions with respect to $\|\cdot\|_\infty$. Since $(C([0,1]), \|\cdot\|_\infty)$ is complete, there exist $f, g \in C([0,1])$ such that $f_n \to f$, $f_n' \to g$ uniformly (i.e. with respect to $\|\cdot\|_\infty$).

- Suppose that (f_n) is a sequence in $C([0,1])$ and $f_n \to f$ uniformly. Let

$$F_n(x) = \int_0^x f_n(t)\, dt, \qquad F(x) = \int_0^x f(t)\, dt.$$

Then $F_n \to F$ uniformly, since

$$\|F_n - F\|_\infty \leq \sup_{x\in[0,1]} \int_0^x |f_n(t) - f(t)|\, dt \leq \|f_n - f\|_\infty .$$

- Since $f_n' \to g$ uniformly, it follows that

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,dt \to \int_0^x g(t)\,dt.$$

  Since $f_n \to f$ uniformly, we conclude that

$$f(x) = f(0) + \int_0^x g(t)\,dt.$$

- The fundamental theorem of calculus implies that f is continuously differentiable and $f' = g$. Thus, $f_n \to f$ and $f_n' \to f'$ uniformly, which implies that $f_n \to f \in C^1([0,1])$ with respect to $\|\cdot\|$. This shows that $(C^1([0,1]), \|\cdot\|)$ is complete.

2. If $f : [0,1] \to \mathbb{R}$ is integrable, define $b_n \in \mathbb{R}$ by

$$b_n = \int_0^1 f(x) \sin(n\pi x)\, dx.$$

(a) Prove that $b_n \to 0$ as $n \to \infty$ for any polynomial.
(b) Prove that $b_n \to 0$ as $n \to \infty$ for any $f \in C([0,1])$.

Solution.

- (a) If f is continuously differentiable, then an integration by parts implies that

$$b_n = -\left[f(x) \frac{\cos(n\pi x)}{n\pi} \right]_0^1 + \frac{1}{n\pi} \int_0^1 f'(x) \cos(n\pi x)\, dx.$$

  Since $|\cos(n\pi x)| \leq 1$, it follows that $b_n \to 0$ as $n \to \infty$.

- (b) Suppose that $f \in C([0,1])$. Given $\epsilon > 0$, there is a continuously differentiable function p such that $\|f - p\|_\infty < \epsilon/2$. (For example, a polynomial; such a polynomial exists by the Weierstrass approximation theorem.)

- By (a), there exists $N \in \mathbb{N}$ such that

$$\left| \int_0^1 p(x) \sin(n\pi x)\, dx \right| < \frac{\epsilon}{2} \qquad \text{for all } n > N.$$

  Then, for $n > N$, we have

$$\begin{aligned} |b_n| &= \left| \int_0^1 f(x) \sin(n\pi x)\, dx \right| \\ &\leq \left| \int_0^1 [f(x) - p(x)] \sin(n\pi x)\, dx \right| + \left| \int_0^1 p(x) \sin(n\pi x)\, dx \right| \\ &\leq \|f - p\|_\infty + \left| \int_0^1 p(x) \sin(n\pi x)\, dx \right| \\ &< \epsilon, \end{aligned}$$

  which proves that $b_n \to 0$ as $n \to \infty$.

3. A function $f : [0,1] \to \mathbb{R}$ is said to be Hölder continuous with exponent α if

$$[f]_\alpha = \sup_{x \neq y \in [0,1]} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}$$

is finite. Given $0 < \alpha \leq 1$ and $M > 0$, define

$$\mathcal{F} = \{ f \in C([0,1]) \mid \|f\|_\infty \leq M, \quad [f]_\alpha \leq M \}.$$

Prove that $\mathcal{F}$ is a compact subset of $C([0,1])$ equipped with the sup-norm $\|\cdot\|_\infty$.

Solution.

- Suppose that $f_n \in \mathcal{F}$ and $f_n \to f$ uniformly. Then

  $$\|f\|_\infty = \lim_{n\to\infty} \|f_n\|_\infty \leq M.$$

  For every $x \neq y \in [0,1]$ and $n \in \mathbb{N}$, we have

  $$\frac{|f_n(x) - f_n(y)|}{|x - y|^\alpha} \leq M.$$

  Taking the limit of this equation as $n \to \infty$, and using the fact that $f_n \to f$ pointwise, we get

  $$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq M,$$

  which implies that $[f]_\alpha \leq M$. Hence $f \in \mathcal{F}$, and $\mathcal{F}$ is closed.

- If f is Hölder continuous and $x \in [0,1]$, then

  $$|f(x) - f(0)| \leq [f]_\alpha |x|^\alpha \leq [f]_\alpha.$$

  Hence,

  $$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq [f]_\alpha + |f(0)|.$$

  Thus, if $f \in \mathcal{F}$, we find that $|f(x)| \leq M + M$, so $\|f\|_\infty \leq 2M$, and therefore $\mathcal{F}$ is bounded.

- Let $\epsilon > 0$. Choose
$$\delta = \left(\frac{\epsilon}{M}\right)^{1/\alpha} > 0.$$
Then if $f \in \mathcal{F}$ and $|x - y| < \delta$, we have
$$|f(x) - f(y)| \leq [f]_\alpha |x - y|^\alpha \leq M|x - y|^\alpha < \epsilon,$$
which shows that $\mathcal{F}$ is (uniformly) equicontinuous.
- The Arzelà-Ascoli theorem implies that $\mathcal{F}$ is a compact.

4. Suppose that (f_n) is a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ such that $|f_n(x)| \leq 1$ for all $n \in \mathbb{N}$, $x \in [0,1]$. Define $F_n : [0,1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Prove that the sequence (F_n) has a subsequence that converges uniformly on $[0,1]$.

Solution.

- We have
$$\|F_n\|_\infty \leq \sup_{x\in[0,1]} \left| \int_0^x f_n(t)\,dt \right| \leq \|f_n\|_\infty \leq 1,$$
so the set $\{F_n \mid n \in \mathbb{N}\}$ is bounded.

- For $x, y \in [0,1]$ and $n \in \mathbb{N}$, we have
$$|F_n(x) - F_n(y)| = \left| \int_x^y f_n(t)\,dt \right| \leq \|f_n\|_\infty |x-y| \leq |x-y|.$$
It follows that $\{F_n \mid n \in \mathbb{N}\}$ is equicontinuous. (We can choose $\delta = \epsilon$ in the definition.)

- By the Arzelà-Ascoli theorem, the family $\{F_n \mid n \in \mathbb{N}\}$ is a precompact subset of $C([0,1])$, so there exists a subsequence (F_{n_k}) that converges uniformly to some function $F \in C([0,1])$.

5. Suppose that
$$\{f_n : K \to \mathbb{R} \mid n \in \mathbb{N}\}$$
is an equicontinuous family of functions on a compact metric space K. If (f_n) converges pointwise to a function f, prove that f is continuous. Is the convergence necessarily uniform?

Solution.

- First, we show that $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ is a bounded subset of $C(K)$.

- The set $\mathcal{F}$ is uniformly equicontinuous since it is equicontinuous and K is compact. Hence, we can choose $\delta > 0$ such that $|f_n(x) - f_n(y)| < 1$ for all $n \in \mathbb{N}$ and $x, y \in K$ such that $d(x, y) < \delta$. Let $\{x_1, \ldots, x_N\}$ be a finite δ-net of K, which exists since K is compact and therefore totally bounded. The sequences $(f_n(x_i))_{n=1}^{\infty}$ converge for $1 \le i \le N$, so they are bounded, by M, say.

- If $x \in K$, then $d(x, x_i) < \delta$ for some $1 \le i \le N$, and therefore
$$|f_n(x)| \le |f_n(x_i)| + |f_n(x) - f_n(x_i)| \le M + 1.$$
It follows that $\|f_n\|_\infty \le M + 1$ for every $n \in \mathbb{N}$, so $\mathcal{F}$ is bounded.

- By the Arzelà-Ascoli theorem, $\mathcal{F}$ is a precompact subset of $C(K)$, so (f_n) has a uniformly convergent subsequence. The limit of this subsequence must be the same as the pointwise limit f of the whole sequence, so f is continuous, since the uniform limit of continuous functions is continuous.

- The sequence (f_n) is contained in a compact set $\overline{\mathcal{F}}$ and every uniformly convergent subsequence has the same limit, namely the pointwise limit f. It follows from the result of Problem 3, Set 4 that the whole sequence converges uniformly to f.