Solutions: Problem Set 7
Math 201A, Fall 2006

Problem 1. Let $r_n = x_{n+1}/x_n$ be the ratio of successive terms in the Fibonacci sequence (x_n) defined by $x_{n+1} = x_n + x_{n-1}$ with $x_0 = x_1 = 1$. Prove that $r_n \to \phi$ as $n \to \infty$ where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

Solution.

- Dividing the recursion relation for the Fibonacci sequence by x_n, we get
$$r_{n+1} = 1 + \frac{1}{r_n}, \qquad r_1 = \frac{3}{2}.$$

- Let $f(r) = 1 + 1/r$. Then $f'(r) = -1/r^2 < 0$ in $r > 0$. Hence $f : (0, \infty) \to \mathbb{R}$ is monotone decreasing. Since $f(3/2) = 5/3$ and $f(2) = 3/2$, we see that $f : [3/2, 2] \to [3/2, 2]$. Moreover $|f'(r)| \leq 4/9$ for $3/2 \leq r \leq 2$, so by the mean value theorem f is a contraction on $[3/2, 2]$. The closed interval $[3/2, 2]$ is complete.

- By the contraction mapping theorem, f has a unique fixed point $\bar{r} \in [3/2, 2]$ and the iterates r_n defined by $r_{n+1} = f(r_n)$ with $r_1 \in [3/2, 2]$ converge to $\bar{r}$ as $n \to \infty$.

- The fixed points $\bar{r}$ of f satisfy $\bar{r} = 1 + 1/\bar{r}$, or $\bar{r}^2 - \bar{r} - 1 = 0$. Hence
$$\bar{r} = \frac{1 \pm \sqrt{5}}{2}.$$
The unique fixed point in $[3/2, 2]$ is ϕ, so $r_n \to \phi$ as $n \to \infty$.

Problem 2. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = 1 + \log(1 + e^x)$$

satisfies $|Tx - Ty| < |x - y|$ for all $x, y \in \mathbb{R}$ with $x \neq y$, but T does not have any fixed points. Why doesn't this example contradict the contraction mapping theorem?

Solution.

- Since
$$T'(x) = \frac{e^x}{1 + e^x},$$
we have $0 < T' < 1$. The mean value theorem implies that if $x \neq y$
$$Tx - Ty = T'(\xi)(x - y)$$
for some ξ between x and y, so
$$|Tx - Ty| < |x - y|.$$

- We have
$$Tx - x = 1 + \log\left(1 + e^{-x}\right) > 1,$$
so T has no fixed point. (In fact, T maps any point a distance greater than 1 away from itself.)

- As $x, y \to \infty$, we have $|Tx - Ty|/|x - y| \to 1$, so there is no constant $c < 1$ such that $|Tx - Ty| \leq c|x - y|$ for all $x, y \in \mathbb{R}$. Thus, T does not satisfy the hypothesis of the contraction mapping theorem.

Problem 3. Suppose that X is a compact metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) < d(x, y) \qquad \text{for all } x, y \in X \text{ with } x \neq y.$$

Prove that T has a unique fixed point in X.

Solution.

- If $Tx = x$, $Ty = y$, and $x \neq y$, then $d(x, y) = d(Tx, Ty) < d(x, y)$, which is a contradiction, so a fixed point of T is unique.

- Define $f : X \to \mathbb{R}$ by $f(x) = d(Tx, x)$. Then f is a continuous function since $T : X \to X$, and $d : X \times X \to \mathbb{R}$ are continuous. The domain X of f is a compact set, so f attains its minimum at some point $\overline{x} \in X$. If $T\overline{x} \neq \overline{x}$, then

$$f(T\overline{x}) = d(T(T\overline{x}), T\overline{x}) < d(T\overline{x}, \overline{x}) = f(\overline{x}),$$

  which contradicts the fact that f attains its minimum at $\overline{x}$. Hence, $\overline{x}$ is a fixed point of T.

Problem 4. Consider the following nonlinear integral equation:

$$f(x) - \frac{1}{\pi}\int_0^1 \frac{f^2(y)}{1+x^2+y^2}\,dy = \frac{3}{4}, \qquad 0 \le x \le 1. \tag{1}$$

Prove that there is a unique continuous solution $f : [0,1] \to [0,1]$ of this equation.

Solution.

- Define T by

  $$(Tf)(x) = \frac{3}{4} + \frac{1}{\pi}\int_0^1 \frac{f^2(y)}{1+x^2+y^2}\,dy.$$

  Then f is a solution of the integral equation if and only if $f = Tf$, meaning that f is a fixed point of T.

- Let

  $$B = \{f \in C([0,1]) \mid f : [0,1] \to [0,1]\}.$$

  Then B is a closed subset of the complete space $C([0,1])$, since pointwise convergence — and hence uniform convergence — preserves the condition $0 \le f(x) \le 1$, so it is complete.

- Suppose that $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. If $f : [0,1] \to \mathbb{R}$ is square-integrable and $g : [0,1] \to \mathbb{R}$ is given by

  $$g(x) = \int_0^1 k(x,y) f^2(y)\,dy,$$

  then

  $$\begin{aligned} |g(x_1) - g(x_2)| &= \left|\int_0^1 [k(x_1,y) - k(x_2,y)]\, f^2(y)\,dy\right| \\ &\le \left(\int_0^1 f^2(y)\,dy\right) \sup_{y\in[0,1]} |k(x_1,y) - k(x_2,y)|. \end{aligned}$$

  Since k is continuous and $[0,1]$ is compact, it follows that g is continuous. In particular, this is true if f is continuous.

- By the previous part, if $f \in C([0,1])$ then $Tf \in C([0,1])$. Since the integrand in (1) is nonnegative, we have $Tf \geq 0$ for every $f \in C[0,1]$. (In fact, $Tf \geq 3/4$.) If $|f| \leq 1$, then

$$\begin{aligned} Tf(x) &\leq \frac{3}{4} + \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2+y^2}\,dy \\ &\leq \frac{3}{4} + \frac{1}{\pi}\int_0^1 \frac{1}{1+y^2}\,dy \\ &\leq 1. \end{aligned}$$

It follows that T maps B into itself.

- We claim that T is a contraction on B. If $f, g \in B$ then $\|f\|_\infty, \|g\|_\infty \leq 1$ and

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left|\frac{1}{\pi}\int_0^1 \frac{f^2(y) - g^2(y)}{1+x^2+y^2}\,dy\right| \\ &\leq \|f^2 - g^2\|_\infty \frac{1}{\pi}\int_0^1 \frac{1}{1+x^2+y^2}\,dy \\ &\leq (\|f\|_\infty + \|g\|_\infty)\,\|f-g\|_\infty \frac{1}{\pi}\int_0^1 \frac{1}{1+y^2}\,dy \\ &\leq \frac{1}{2}\|f-g\|_\infty. \end{aligned}$$

Hence, $\|Tf - Tg\|_\infty \leq c\|f-g\|_\infty$ with $c = 1/2$.

- By the contraction mapping theorem, T has a unique fixed point $f \in B$, which proves the result.