Solutions: Problem Set 8
Math 201A, Fall 2006

Problem 1. Recall that a function $f : [0, 1] \to \mathbb{R}$ is Lipschitz continuous if its Lipschitz constant

$$\mathrm{Lip}(f) = \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|}$$

is finite.
(a) For $M > 0$, let

$$L_M = \{f \in C([0, 1]) \mid \mathrm{Lip}(f) \leq M\}.$$

Show that L_M is a closed subset of $C([0, 1])$ equipped with the sup-norm.
(b) Let $L = \{f \in C([0, 1]) \mid f \text{ is Lipschitz continuous}\}$. Prove that L is a linear subspace of $C([0, 1])$.
(c) Is L a closed linear subspace of $C([0, 1])$ equipped with the sup-norm?

Solution.

- (a) Suppose that $f_n \in L_M$ and $f_n \to f$ uniformly. For every $x, y \in [0, 1]$ and $n \in \mathbb{N}$, we have
$$\frac{|f_n(x) - f_n(y)|}{|x - y|} \leq M.$$
Taking the limit of this equation as $n \to \infty$, and using the fact that $f_n \to f$ pointwise, we get
$$\frac{|f(x) - f(y)|}{|x - y|} \leq M,$$
which implies that $f \in L_M$. Thus, L_M is closed.

- (b) If λ is a scalar, then
$$|(\lambda f)(x) - (\lambda f)(y)| = |\lambda||f(x) - f(y)|.$$
It follows that if f is Lipschitz continuous, then λf is Lipschitz continuous, and $\mathrm{Lip}(\lambda f) = |\lambda|\mathrm{Lip}(f)$. If f, g are Lipschitz continuous, with
$$|f(x) - f(y)| \leq M|x - y|, \qquad |g(x) - g(y)| \leq N|x - y|,$$

then

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq (M+N)|x-y|. \end{aligned}$$

Hence $(f+g)$ is Lipschitz continuous and $\text{Lip}(f+g) \leq \text{Lip}(f) + \text{Lip}(g)$. It follows that L is a linear space.

- (c) Any polynomial is Lipschitz continuous by the mean value theorem (since it is differentiable with bounded derivative on $[0,1]$). By the Weierstrass approximation theorem, the closure of L is $C([0,1])$, so L is not closed.

- Alternatively, one can give an explicit sequence (f_n) of Lipschitz continuous functions that converges uniformly to a non-Lipschitz continuous function; for example, $f_n(x) = (x + 1/n)^{1/2}$.

Problem 2. Let $c_0(\mathbb{N})$ be the Banach space of real sequences $x = (x_1, x_2, x_3, \ldots)$ such that $x_n \to 0$ as $n \to \infty$, equipped with the sup norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|.$$

(a) Let $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ be the sequence with nth term equal to 1 and all other terms equal to 0. Show that $(e_n)_{n=1}^{\infty}$ is a Schauder basis of $c_0(\mathbb{N})$.
(b) Let

$$f_1 = \frac{1}{2}e_1, \qquad f_n = \frac{1}{2}e_n - e_{n-1} \quad \text{for } n \geq 2.$$

Show that $\{f_n \mid n \in \mathbb{N}\}$ is a linearly independent set. Is $(f_n)_{n=1}^{\infty}$ a Schauder basis of $c_0(\mathbb{N})$?

Solution.

- (a) If $x \in c_0$, then

$$\left\| x - \sum_{n=1}^{N} x_n e_n \right\| = \sup_{n\geq N+1} |x_n| \to 0 \qquad \text{as } N \to \infty,$$

  so $x = \sum_{n=1}^{\infty} x_n e_n$.

- If $x = \sum_{n=1}^{\infty} c_n e_n$, then

$$|x_n - c_n| \leq \lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} c_n e_n \right\| = 0,$$

  so $c_n = x_n$, and this expansion is unique. Hence (e_n) is a Schauder basis of c_0.

- (b) If $c_1 f_1 + c_2 f_2 + \ldots + c_N f_N = 0$, then

$$\left(\frac{1}{2}c_1 - c_2\right) e_1 + \left(\frac{1}{2}c_2 - c_3\right) e_2 + \ldots + \left(\frac{1}{2}c_{N-1} - c_N\right) e_{N-1} + \frac{1}{2}c_N e_N = 0.$$

  Since $\{e_n\}$ is a linearly independent set, it follows that

$$\frac{1}{2}c_1 - c_2 = 0, \quad \frac{1}{2}c_2 - c_3 = 0, \quad \frac{1}{2}c_{N-1} - c_N = 0, \quad \frac{1}{2}c_N = 0,$$

  so $c_n = 0$ for $1 \leq n \leq N$, and $\{f_n\}$ is a linearly independent set.

- The sequence (f_n) is not a Schauder basis since an expansion with respect to this sequence is not unique. For example,

$$\sum_{n=1}^{N} \frac{1}{2^n} f_n = \frac{1}{2^{N+2}} e_{N+1} \to 0 \qquad \text{as } N \to \infty,$$

so for any $c \in \mathbb{R}$, we have

$$0 = \sum_{n=1}^{\infty} \frac{c}{2^n} f_n.$$

Problem 3. Suppose that X, Y, Z are normed linear spaces and $A : X \to Y$, $B : Y \to Z$ are bounded linear operators. Prove that $BA : X \to Z$ is a bounded linear operator, and

$$\|BA\| \leq \|A\|\|B\|.$$

Give an example to show that this inequality may be strict.

Solution.

- Using the definitions of $\|B\|$ and $\|A\|$, we have

$$\begin{aligned} \|BAx\| &= \|B(Ax)\| \\ &\leq \|B\| \, \|Ax\| \\ &\leq \|B\| \, \|A\| \, \|x\|. \end{aligned}$$

  It follows that BA is bounded and $\|BA\| \leq \|B\|\|A\|$.

- Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear maps with matrices

$$[A] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad [B] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

  Then $\|A\| = \|B\| = 1$ with respect to any norm on $\mathbb{R}^2$, but $BA = 0$, so $\|BA\| = 0$.

Problem 4. Let $\delta : C([0,1]) \to \mathbb{R}$ be the functional that evaluates a function at the origin, defined by $\delta(f) = f(0)$.
(a) Show that δ is a linear functional. What is the kernel of δ? What is the range of δ?
(b) If $C([0,1])$ is equipped with the sup-norm

$$\|f\|_\infty = \sup_{0 \le x \le 1} |f(x)|,$$

show that δ is bounded and compute its norm.
(c) If $C([0,1])$ is equipped with the 1-norm

$$\|f\|_1 = \int_0^1 |f(x)|\, dx,$$

show that δ is unbounded.
(d) Give an example of a bounded linear functional $F : C([0,1]) \to \mathbb{R}$ where $C([0,1])$ is equipped with the 1-norm and compute its norm.

Solution.

- (a) We have $\delta(\lambda f) = \lambda f(0) = \lambda \delta(f)$ and

  $$\delta(f+g) = (f+g)(0) = f(0) + g(0) = \delta(f) + \delta(g),$$

  so δ is a linear functional.

- The kernel of δ is $\{f \in C([0,1]) \mid f(0) = 0\}$ and the range of δ is $\mathbb{R}$.

- (b) We have $|\delta(f)| = |f(0)| \le \|f\|_\infty$, so δ is bounded and $\|\delta\| \le 1$. Also, $|\delta(1)| = \|1\|_\infty$, so $\|\delta\| \ge 1$, and hence $\|\delta\| = 1$.

- (c) If $f_n(x) = e^{-nx}$, then $|\delta(f_n)| = 1$ and $\|f_n\|_1 < 1/n$. Hence δ is unbounded, since

  $$\|\delta\| \ge \sup_{n \in \mathbb{N}} \frac{|\delta(f_n)|}{\|f_n\|} = \infty.$$

- (d) Define a linear functional $M : C([0,1]) \to \mathbb{R}$ by

  $$Mf = \int_0^1 f(x)\, dx.$$

  Then $|Mf| \le \|f\|_1$ and $|M1| = \|1\|_1$, so M is bounded with $\|M\| = 1$.