Solutions: Problem Set 9
Math 201A, Fall 2006

Problem 1. Let c be the Banach space of all convergent real sequences $(x_n)_{n=1}^{\infty}$, and c_0 the subspace of sequences that converge to 0, both equipped with the ∞-norm, $\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
(a) Define $L : c \to \mathbb{R}$ by $L(x_n) = \lim_{n\to\infty} x_n$. Prove that L is a bounded linear functional on c and compute its norm.
(b) For $x = (x_n) \in c$, define $Tx = y$ where $y = (y_n)_{n=1}^{\infty}$ is given by

$$y_1 = Lx, \qquad y_{n+1} = x_n - Lx \quad \text{for } n \geq 1.$$

Prove that $T : c \to c_0$ is a one-to-one, onto bounded linear map. (It follows from the open mapping theorem that T^{-1} is bounded, so c_0 and c are topologically isomorphic Banach spaces.)

Solution.

- (a) The functional L is linear, since if $x = (x_n) \in c$, $\tilde{x} = (\tilde{x}_n) \in c$ and $\lambda \in \mathbb{R}$, then

$$L(\lambda x) = \lim_{n\to\infty} \lambda x_n = \lambda \lim_{n\to\infty} x_n = \lambda L x,$$
$$L(x + \tilde{x}) = \lim_{n\to\infty} (x_n + \tilde{x}_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} \tilde{x}_n = Lx + L\tilde{x}.$$

- We have

$$|L(x_n)| = |\lim_{n\to\infty} x_n| = \lim_{n\to\infty} |x_n| \leq \sup_{n\in\mathbb{N}} |x_n| = \|(x_n)\|_\infty,$$

  so L is bounded, with $\|L\| \leq 1$.

- Conversely, if $x_n = 1$ for every $n \in \mathbb{N}$, then $(x_n) \in c$ with $L(x_n) = 1$ and $\|(x_n)\| = 1$, so $\|L\| \geq 1$. It follows that $\|L\| = 1$.

- (b) It follows from the definition of T that if $x \in c$ and $y = Tx$, then $\lim_{n\to\infty} y_n = 0$ so $y \in c_0$. Therefore $T : c \to c_0$. It is easy to check that T is linear.

- If $Tx = T\tilde{x}$, then $Lx = L\tilde{x}$ and $x_n - Lx = \tilde{x}_n - L\tilde{x}$ for $n \geq 1$, so $x_n = \tilde{x}_n$, and $x = \tilde{x}$. Hence T is one-to-one.

- If $y = (y_n) \in c_0$, define the sequence $x = (x_n)$ by

$$x_n = y_{n+1} + y_1 \qquad n \geq 1.$$

  Then $x \in c$, with $Lx = y_1$, and $Tx = y$. Hence T maps c onto c_0.

- If $x = (x_n) \in c$, then using the fact that $\|L\| = 1$ we have

$$\begin{aligned} \|Tx\| &= \sup\{|Lx|, |x_1 - Lx|, |x_2 - Lx|, \ldots\} \\ &\leq \sup\{|Lx|, |x_1| + |Lx|, |x_2| + |Lx|, \ldots\} \\ &\leq |Lx| + \sup\{|x_1|, |x_2|, \ldots\} \\ &\leq \|L\|\,\|x\| + \|x\| \\ &\leq 2\|x\|. \end{aligned}$$

  Thus, T is bounded and $\|T\| \leq 2$.

- Although this was not asked, note that if $x = (-1, 1, 1, 1, \ldots)$ then $Tx = (1, -2, 0, 0, \ldots)$. Since $\|x\| = 1$ and $\|Tx\| = 2$, we see that $\|T\| = 2$.

Remark. Although c_0 and c are topologically isomorphic they are not isometrically isomorphic (see Problem 4 in the Fall 2004 Final Exam).

Problem 2. A sequence (T_n) of bounded linear operators $T_n : X \to Y$ on normed linear spaces X, Y is said to converge strongly to $T : X \to Y$ if $T_n x \to Tx$ in norm in Y for every $x \in X$.

(a) Show that if $T_n \to T$ uniformly (i.e. with respect to the operator norm), then $T_n \to T$ strongly.

(b) Let $C_0(\mathbb{R})$ be the Banach space of continuous functions that approach zero at ∞, equipped with the sup-norm. For $h \in \mathbb{R}$ define the translation operator $T_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$T_h f(x) = f(x+h).$$

Prove that $T_h \to I$ strongly as $h \to 0$, where I is the identity operator on $C_0(\mathbb{R})$. Prove that T_h does not converge to I uniformly as $h \to 0$.

(c) With T_h as in (b), define $A_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by

$$A_h = \frac{T_h - I}{h}.$$

Does A_h converge strongly as $h \to 0$? For what $f \in C_0(\mathbb{R})$ does $A_h f$ converge in norm as $h \to 0$? Compute the limit when it exists.

Solution.

- (a) For any $x \in X$, we have

  $$\|T_n x - Tx\| \leq \|T_n - T\| \, \|x\|.$$

  It follows that if $T_n \to T$ with respect to the operator norm, then $T_n x \to Tx$ with respect to the norm on Y. Thus $T_n \to T$ uniformly implies that $T_n \to T$ strongly.

- (b) Suppose that $f \in C_0(\mathbb{R})$. Let $\epsilon > 0$. Since $f(x) \to 0$ as $|x| \to \infty$, there exists $R > 0$ such that

  $$|f(x)| < \frac{\epsilon}{2} \qquad \text{for } |x| \geq R.$$

  Since f is continuous, it is uniformly continuous on any compact interval, and there exists $0 < \delta \leq 1$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in [-R-1, R+1]$ with $|x-y| < \delta$. It follows that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$ such that $|x-y| < \delta$, meaning that f is uniformly continuous on $\mathbb{R}$.

- If $|h| < \delta$, then $|f(x+h) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$, and

$$\|T_h f - f\|_\infty = \sup_{x \in \mathbb{R}} |f(x+h) - f(x)| < \epsilon.$$

Hence, $T_h f \to f$ uniformly as $h \to 0$ for every $f \in C_0(\mathbb{R})$, meaning that T_h converges strongly to I.

- Given any $h > 0$, consider $f \in C_0(\mathbb{R})$ such that $\|f\|_\infty = 1$ and the support of f is contained in the interval $(0, h)$. Then $T_h f$ and f have disjoint supports and $\|T_h f - f\|_\infty = 1$. It follows that $\|T_h - I\| \geq 1$, so T_h does not converge to I with respect to the operator norm.

- (c) If $A_h f$ converges as $h \to 0$ with respect to the sup-norm, then the pointwise limit

$$\lim_{h \to 0} A_h f(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

exists for every $x \in \mathbb{R}$, so f is differentiable on $\mathbb{R}$. Furthermore, the derivative f' is a uniform limit of functions in $C_0(\mathbb{R})$, so $f' \in C_0(\mathbb{R})$ since $C_0(\mathbb{R})$ is closed with respect to uniform convergence. Thus, $A_h f$ does not converge with respect to the sup-norm for any $f \in C_0(\mathbb{R})$ that is not differentiable on $\mathbb{R}$ with $f' \in C_0(\mathbb{R})$. Hence A_h does not converge strongly on $C_0(\mathbb{R})$ as $h \to 0$.

- If f is continuously differentiable, then the fundamental theorem of calculus implies that

$$A_h f(x) = \frac{1}{h} \int_x^{x+h} f'(t)\, dt.$$

It follows that

$$A_h f(x) - f'(x) = \frac{1}{h} \int_x^{x+h} \left[f'(t) - f'(x) \right] dt,$$

where we have written

$$f'(x) = \frac{1}{h} \int_x^{x+h} f'(x)\, dt.$$

- If $f' \in C_0(\mathbb{R})$, then the uniform continuity of f' implies that given $\epsilon > 0$ there exists $\delta > 0$ such that $|f'(x) - f'(y)| < \epsilon$ if $|x - y| < \delta$. Hence, if $|h| < \delta$, we have

$$|A_h f(x) - f'(x)| \leq \frac{1}{h} \int_x^{x+h} |f'(t) - f'(x)| \, dt < \epsilon.$$

  It follows that $\|A_h f - f\|_\infty < \epsilon$ if $|h| < \delta$, which proves that $A_h f \to f$ uniformly for $f \in C_0(\mathbb{R})$ if and only if $f' \in C_0(\mathbb{R})$.

Remark. It is not quite correct to say that $A_h f$ converges uniformly if $f \in C_0(\mathbb{R})$ is continuously differentiable on $\mathbb{R}$, since the derivative need not approach 0 at infinity; for example, consider

$$f(x) = \frac{\sin x^3}{1 + x^2}.$$

Problem 3. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with $m \times n$ matrix (a_{ij}) with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^m$.

(a) Compute the operator (or matrix) norm $\|A\|$ if the domain $\mathbb{R}^n$ is equipped with the 1-norm,

$$\|(x_1, \dots, x_n)\|_1 = |x_1| + \dots + |x_n|,$$

and the range $\mathbb{R}^m$ is equipped with the ∞-norm,

$$\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

(b) If the domain $\mathbb{R}^n$ is equipped with the ∞-norm and the range $\mathbb{R}^m$ is equipped with the 1-norm, prove that

$$\|A\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|,$$

with equality if $a_{ij} \geq 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Solution.

- (a) We have

$$\begin{aligned}
\|Ax\|_\infty &= \max_{1\leq i\leq m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \\
&\leq \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} |a_{ij}| \, |x_j| \right\} \\
&\leq \max_{1\leq i\leq m} \left\{ \max_{1\leq j\leq n} \{|a_{ij}|\} \sum_{j=1}^{n} |x_j| \right\} \\
&\leq \max\{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\} \|x\|_1.
\end{aligned}$$

  It follows that the operator norm of

$$A : (\mathbb{R}^n, \|\cdot\|_1) \to (\mathbb{R}^m, \|\cdot\|_\infty)$$

  satisfies

$$\|A\| \leq \max\{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

- Choose $1 \leq I \leq m$, $1 \leq J \leq n$ such that

$$|a_{IJ}| = \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Define $x = (x_j) \in \mathbb{R}^n$ by $x_j = 0$ if $j \neq J$ and

$$x_J = \begin{cases} 1 & \text{if } a_{IJ} > 0, \\ -1 & \text{if } a_{IJ} < 0. \end{cases}$$

(If $a_{IJ} = 0$, then $A = 0$ and $\|A\| = 0$.) Then $\|x\|_1 = 1$ and $\|Ax\|_\infty = |a_{IJ}|$, so $\|A\| \geq |a_{IJ}|$. It follows that

$$\|A\| = \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

- (b) We have

$$\begin{aligned} \|Ax\|_1 &= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \\ &\leq \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \right\} \max_{1 \leq j \leq n} |x_j| \\ &\leq \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \right\} \|x\|_\infty. \end{aligned}$$

It follows that the operator norm of

$$A : (\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^m, \|\cdot\|_1)$$

satisfies

$$\|A\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$

- If the matrix of A has nonnegative coefficients and $x = (1, 1, \ldots, 1)$, then $\|x\|_\infty = 1$ and

$$\|Ax\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

It follows that

$$\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

Problem 4. Let $\ell^2(\mathbb{N})$ be the Banach space of square-summable real sequences $x = (x_i)_{i=1}^\infty$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}.$$

A sequence $(x^{(n)})$ in $\ell^2(\mathbb{N})$,

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots),$$

converges weakly to $x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$ as $n \to \infty$ if for every

$$y = (y_1, y_2, y_3, \dots) \in \ell^2(\mathbb{N})$$

we have

$$\sum_{i=1}^{\infty} x_i^{(n)} y_i \to \sum_{i=1}^{\infty} x_i y_i \quad \text{as } n \to \infty.$$

Let $e^{(n)} = (0, 0, \dots 0, 1, 0, \dots)$ be the element of $\ell^2(\mathbb{N})$ with $x_i = 1$ when $i = n$ and $x_i = 0$ otherwise.

(a) Prove that the sequence $(e^{(n)})$ converges weakly to 0 as $n \to \infty$, but does not converge strongly (i.e. in norm) to any limit.

(b) Does the sequence $(ne^{(n)})$ converge weakly as $n \to \infty$?

Solution.

- We use the notation
$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$
for the inner-product $\langle \cdot, \cdot \rangle : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R}$. According to the Cauchy-Schwarz inequality $|\langle x, y \rangle| \le \|x\| \, \|y\|$, so the inner product $\langle x, y \rangle$ is well-defined and finite for every $x, y \in \ell^2(\mathbb{N})$.

- Weak convergence $x^{(n)} \rightharpoonup x$ in $\ell^2(\mathbb{N})$ then means that
$$\langle x^{(n)}, y \rangle \to \langle x, y \rangle \qquad \text{as } n \to \infty \text{ for every } y \in \ell^2(\mathbb{N}).$$

- (a) If $y = (y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N})$, then

  $$\sum_{i=1}^{\infty} |y_i|^2 < \infty,$$

  so $y_i \to 0$ as $i \to \infty$.

- For any $y \in \ell^2(\mathbb{N})$, we have

  $$\langle e^{(n)}, y \rangle = y_n \to 0 = \langle 0, y \rangle \qquad \text{as } n \to \infty,$$

  which means that $e^{(n)}$ converges weakly to 0 as $n \to \infty$.

- For every $m \neq n$, we have $\|e^{(m)} - e^{(n)}\| = \sqrt{2}$, so the sequence $(e^{(n)})$ is not Cauchy and therefore does not converge in norm.

- (b) The sequence $(ne^{(n)})$ does not converge weakly. For example, consider $y = (y_i)$ defined by

  $$y_i = \frac{1}{i^{3/4}}.$$

  Then $y \in \ell^2(\mathbb{N})$, since

  $$\sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty,$$

  but

  $$\langle ne^{(n)}, y \rangle = n^{1/4},$$

  does not converge as $n \to \infty$. Hence, there is no $x \in \ell^2(\mathbb{N})$ such that $\langle ne^{(n)}, y \rangle \to \langle x, y \rangle$ as $n \to \infty$.

Remark. It is a consequence of the uniform boundedness theorem that any weakly convergent sequence is bounded in norm, as is true of the sequence in (a) but not of the sequence in (b).