Problem 1. Let \( c \) be the Banach space of all convergent real sequences \((x_n)_{n=1}^{\infty}\), and \( c_0 \) the subspace of sequences that converge to 0, both equipped with the \( \infty \)-norm, \( \|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} |x_n| \).

(a) Define \( L : c \to \mathbb{R} \) by \( L(x_n) = \lim_{n \to \infty} x_n \). Prove that \( L \) is a bounded linear functional on \( c \) and compute its norm.

(b) For \( x = (x_n) \in c \), define \( T x = y \) where \( y = (y_n)_{n=1}^{\infty} \) is given by
\[
y_1 = Lx, \quad y_{n+1} = x_n - Lx \quad \text{for } n \geq 1.
\]
Prove that \( T : c \to c_0 \) is a one-to-one, onto bounded linear map. (It follows from the open mapping theorem that \( T^{-1} \) is bounded, so \( c_0 \) and \( c \) are topologically isomorphic Banach spaces.)

Solution.

• (a) The functional \( L \) is linear, since if \( x = (x_n) \in c \), \( \hat{x} = (\hat{x}_n) \in c \) and \( \lambda \in \mathbb{R} \), then
\[
L(\lambda x) = \lim_{n \to \infty} \lambda x_n = \lambda \lim_{n \to \infty} x_n = \lambda Lx,
\]
\[
L(x + \hat{x}) = \lim_{n \to \infty} (x_n + \hat{x}_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} \hat{x}_n = Lx + L\hat{x}.
\]

• We have
\[
|L(x_n)| = |\lim_{n \to \infty} x_n| = \lim_{n \to \infty} |x_n| \leq \sup_{n \in \mathbb{N}} |x_n| = \|(x_n)\|_\infty,
\]
so \( L \) is bounded, with \( \|L\| \leq 1 \).

• Conversely, if \( x_n = 1 \) for every \( n \in \mathbb{N} \), then \((x_n) \in c \) with \( L(x_n) = 1 \) and \( \|(x_n)\|_\infty = 1 \), so \( \|L\| \geq 1 \). It follows that \( \|L\| = 1 \).

• (b) It follows from the definition of \( T \) that if \( x \in c \) and \( y = Tx \), then \( \lim_{n \to \infty} y_n = 0 \) so \( y \in c_0 \). Therefore \( T : c \to c_0 \). It is easy to check that \( T \) is linear.

• If \( Tx = T\hat{x} \), then \( Lx = L\hat{x} \) and \( x_n - Lx = \hat{x}_n - L\hat{x} \) for \( n \geq 1 \), so \( x_n = \hat{x}_n \), and \( x = \hat{x} \). Hence \( T \) is one-to-one.
• If $y = (y_n) \in c_0$, define the sequence $x = (x_n)$ by
  \[ x_n = y_{n+1} + y_1 \quad n \geq 1. \]
Then $x \in c$, with $Lx = y_1$, and $Tx = y$. Hence $T$ maps $c$ onto $c_0$.

• If $x = (x_n) \in c$, then using the fact that $\|L\| = 1$ we have
  \[
  \|Tx\| = \sup\{ |Lx|, |x_1 - Lx|, |x_2 - Lx|, \ldots \} \\
  \leq \sup\{ |Lx|, |x_1| + |Lx|, |x_2| + |Lx|, \ldots \} \\
  \leq |Lx| + \sup\{ |x_1|, |x_2|, \ldots \} \\
  \leq \|L\| \|x\| + \|x\| \\
  \leq 2\|x\|.
  \]
Thus, $T$ is bounded and $\|T\| \leq 2$.

• Although this was not asked, note that if $x = (-1, 1, 1, \ldots)$ then $Tx = (1, -2, 0, 0, \ldots)$. Since $\|x\| = 1$ and $\|Tx\| = 2$, we see that $\|T\| = 2$.

Remark. Although $c_0$ and $c$ are topologically isomorphic they are not isometrically isomorphic (see Problem 4 in the Fall 2004 Final Exam).

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**Problem 2.** A sequence \((T_n)\) of bounded linear operators \(T_n : X \to Y\) on normed linear spaces \(X, Y\) is said to converge strongly to \(T : X \to Y\) if \(T_n x \to Tx\) in norm in \(Y\) for every \(x \in X\).

(a) Show that if \(T_n \to T\) uniformly (i.e. with respect to the operator norm), then \(T_n \to T\) strongly.

(b) Let \(C_0(\mathbb{R})\) be the Banach space of continuous functions that approach zero at \(\infty\), equipped with the sup-norm. For \(h \in \mathbb{R}\) define the translation operator \(T_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})\) by
\[
T_h f(x) = f(x + h).
\]
Prove that \(T_h \to I\) strongly as \(h \to 0\), where \(I\) is the identity operator on \(C_0(\mathbb{R})\). Prove that \(T_h\) does not converge to \(I\) uniformly as \(h \to 0\).

(c) With \(T_h\) as in (b), define \(A_h : C_0(\mathbb{R}) \to C_0(\mathbb{R})\) by
\[
A_h = \frac{T_h - I}{h}.
\]
Does \(A_h\) converge strongly as \(h \to 0\)? For what \(f \in C_0(\mathbb{R})\) does \(A_h f\) converge in norm as \(h \to 0\)? Compute the limit when it exists.

**Solution.**

- (a) For any \(x \in X\), we have
\[
\|T_n x - Tx\| \leq \|T_n - T\| \|x\|.
\]
It follows that if \(T_n \to T\) with respect to the operator norm, then \(T_n x \to Tx\) with respect to the norm on \(Y\). Thus \(T_n \to T\) uniformly implies that \(T_n \to T\) strongly.

- (b) Suppose that \(f \in C_0(\mathbb{R})\). Let \(\epsilon > 0\). Since \(f(x) \to 0\) as \(|x| \to \infty\), there exists \(R > 0\) such that
\[
|f(x)| < \frac{\epsilon}{2} \quad \text{for} \quad |x| \geq R.
\]
Since \(f\) is continuous, it is uniformly continuous on any compact interval, and there exists \(0 < \delta \leq 1\) such that \(|f(x) - f(y)| < \epsilon\) for all \(x, y \in [-R-1, R+1]\) with \(|x - y| < \delta\). It follows that \(|f(x) - f(y)| < \epsilon\) for all \(x, y \in \mathbb{R}\) such that \(|x - y| < \delta\), meaning that \(f\) is uniformly continuous on \(\mathbb{R}\).
• If $|h| < \delta$, then $|f(x + h) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$, and

$$\|T_h f - f\|_\infty = \sup_{x \in \mathbb{R}} |f(x + h) - f(x)| < \epsilon.$$ 

Hence, $T_h f \to f$ uniformly as $h \to 0$ for every $f \in C_0(\mathbb{R})$, meaning that $T_h$ converges strongly to $I$.

• Given any $h > 0$, consider $f \in C_0(\mathbb{R})$ such that $\|f\|_\infty = 1$ and the support of $f$ is contained in the interval $(0, h)$. Then $T_h f$ and $f$ have disjoint supports and $\|T_h f - f\|_\infty = 1$. It follows that $\|T_h - I\| \geq 1$, so $T_h$ does not converge to $I$ with respect to the operator norm.

• (c) If $A_h f$ converges as $h \to 0$ with respect to the sup-norm, then the pointwise limit

$$\lim_{h \to 0} A_h f(x) = \lim_{h \to 0} \left[ \frac{f(x + h) - f(x)}{h} \right]$$

exists for every $x \in \mathbb{R}$, so $f$ is differentiable on $\mathbb{R}$. Furthermore, the derivative $f'$ is a uniform limit of functions in $C_0(\mathbb{R})$, so $f' \in C_0(\mathbb{R})$ since $C_0(\mathbb{R})$ is closed with respect to uniform convergence. Thus, $A_h f$ does not converge with respect to the sup-norm for any $f \in C_0(\mathbb{R})$ that is not differentiable on $\mathbb{R}$ with $f' \in C_0(\mathbb{R})$. Hence $A_h$ does not converge strongly on $C_0(\mathbb{R})$ as $h \to 0$.

• If $f$ is continuously differentiable, then the fundamental theorem of calculus implies that

$$A_h f(x) = \frac{1}{h} \int_x^{x+h} f'(t) \, dt.$$ 

It follows that

$$A_h f(x) - f'(x) = \frac{1}{h} \int_x^{x+h} [f'(t) - f'(x)] \, dt,$$

where we have written

$$f'(x) = \frac{1}{h} \int_x^{x+h} f'(x) \, dt.$$
• If $f' \in C_0(\mathbb{R})$, then the uniform continuity of $f'$ implies that given $\epsilon > 0$ there exists $\delta > 0$ such that $|f'(x) - f'(y)| < \epsilon$ if $|x - y| < \delta$. Hence, if $|h| < \delta$, we have

$$|A_h f(x) - f'(x)| \leq \frac{1}{h} \int_x^{x+h} |f'(t) - f'(x)| \, dt < \epsilon.$$ 

It follows that $\|A_h f - f\|_\infty < \epsilon$ if $|h| < \delta$, which proves that $A_h f \to f$ uniformly for $f \in C_0(\mathbb{R})$ if and only if $f' \in C_0(\mathbb{R})$.

**Remark.** It is not quite correct to say that $A_h f$ converges uniformly if $f \in C_0(\mathbb{R})$ is continuously differentiable on $\mathbb{R}$, since the derivative need not approach 0 at infinity; for example, consider

$$f(x) = \frac{\sin x^3}{1 + x^2}.$$
Problem 3. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with $m \times n$ matrix $(a_{ij})$ with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^m$.
(a) Compute the operator (or matrix) norm $\|A\|$ if the domain $\mathbb{R}^n$ is equipped with the 1-norm,

$$\|(x_1, \ldots, x_n)\|_1 = |x_1| + \ldots + |x_n|,$$

and the range $\mathbb{R}^m$ is equipped with the $\infty$-norm,

$$\|(x_1, \ldots, x_n)\|_\infty = \max \{|x_1|, \ldots, |x_n|\}.$$

(b) If the domain $\mathbb{R}^n$ is equipped with the $\infty$-norm and the range $\mathbb{R}^m$ is equipped with the 1-norm, prove that

$$\|A\| \leq \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|,$$

with equality if $a_{ij} \geq 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Solution.

- (a) We have

$$\|Ax\|_\infty = \max_{1 \leq i \leq m} \left| \sum_{j=1}^n a_{ij} x_j \right|$$

$$\leq \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^n |a_{ij}| |x_j| \right\}$$

$$\leq \max_{1 \leq i \leq m} \left\{ \max_{1 \leq j \leq n} \{|a_{ij}|\} \sum_{j=1}^n |x_j| \right\}$$

$$\leq \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\} \|x\|_1.$$

It follows that the operator norm of

$$A : (\mathbb{R}^n, \cdot \|\|_1) \to (\mathbb{R}^m, \cdot \|\|_\infty)$$

satisfies

$$\|A\| \leq \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$
• Choose $1 \leq I \leq m$, $1 \leq J \leq n$ such that
  
  $$|a_{IJ}| = \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$ 

  Define $x = (x_j) \in \mathbb{R}^n$ by $x_j = 0$ if $j \neq J$ and
  
  $$x_j = \begin{cases} 
  1 & \text{if } a_{IJ} > 0, \\
  -1 & \text{if } a_{IJ} < 0.
  \end{cases}$$

  (If $a_{IJ} = 0$, then $A = 0$ and $\|A\| = 0$.) Then $\|x\|_1 = 1$ and $\|Ax\|_{\infty} = |a_{IJ}|$, so $\|A\| \geq |a_{IJ}|$. It follows that
  
  $$\|A\| = \max \{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$ 

• (b) We have
  
  $$\|Ax\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

  $$\leq \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \right\} \max_{1 \leq j \leq n} |x_j|$$

  $$\leq \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \right\} \|x\|_{\infty}.$$ 

  It follows that the operator norm of
  
  $$A : (\mathbb{R}^n, \| \cdot \|_{\infty}) \to (\mathbb{R}^m, \| \cdot \|_1)$$

  satisfies
  
  $$\|A\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.$$ 

• If the matrix of $A$ has nonnegative coefficients and $x = (1,1,\ldots,1)$, then $\|x\|_{\infty} = 1$ and
  
  $$\|Ax\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$ 

  It follows that
  
  $$\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$
Problem 4. Let $\ell^2(\mathbb{N})$ be the Banach space of square-summable real sequences $x = (x_i)_{i=1}^{\infty}$ with norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}.$$ 

A sequence $(x^{(n)})$ in $\ell^2(\mathbb{N})$, 

$$x^{(n)} = (x^{(n)}_1, x^{(n)}_2, x^{(n)}_3, \ldots),$$

converges weakly to $x = (x_1, x_2, x_3, \ldots) \in \ell^2(\mathbb{N})$ as $n \to \infty$ if for every 

$$y = (y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N})$$

we have 

$$\sum_{i=1}^{\infty} x^{(n)}_i y_i \to \sum_{i=1}^{\infty} x_i y_i \quad \text{as} \quad n \to \infty.$$ 

Let $e^{(n)} = (0, 0, \ldots, 0, 1, 0, \ldots)$ be the element of $\ell^2(\mathbb{N})$ with $x_i = 1$ when $i = n$ and $x_i = 0$ otherwise.

(a) Prove that the sequence $(e^{(n)})$ converges weakly to $0$ as $n \to \infty$, but does not converge strongly (i.e. in norm) to any limit.

(b) Does the sequence $(ne^{(n)})$ converge weakly as $n \to \infty$?

Solution.

- We use the notation 

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

for the inner-product $\langle \cdot, \cdot \rangle : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R}$. According to the Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$, so the inner product $\langle x, y \rangle$ is well-defined and finite for every $x, y \in \ell^2(\mathbb{N})$.

- Weak convergence $x^{(n)} \rightharpoonup x$ in $\ell^2(\mathbb{N})$ then means that 

$$\langle x^{(n)}, y \rangle \to \langle x, y \rangle \quad \text{as} \quad n \to \infty \quad \text{for every} \quad y \in \ell^2(\mathbb{N}).$$
• (a) If \( y = (y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N}) \), then
\[
\sum_{i=1}^{\infty} |y_i|^2 < \infty,
\]
so \( y_i \to 0 \) as \( i \to \infty \).

• For any \( y \in \ell^2(\mathbb{N}) \), we have
\[
\langle e^{(n)}, y \rangle = y_n \to 0 = \langle 0, y \rangle \quad \text{as } n \to \infty,
\]
which means that \( e^{(n)} \) converges weakly to 0 as \( n \to \infty \).

• For every \( m \neq n \), we have \( \|e^{(m)} - e^{(n)}\| = \sqrt{2} \), so the sequence \( (e^{(n)}) \) is not Cauchy and therefore does not converge in norm.

• (b) The sequence \( (ne^{(n)}) \) does not converge weakly. For example, consider \( y = (y_i) \) defined by
\[
y_i = \frac{1}{i^{3/4}}.
\]
Then \( y \in \ell^2(\mathbb{N}) \), since
\[
\sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty,
\]
but
\[
\langle ne^{(n)}, y \rangle = n^{1/4},
\]
does not converge as \( n \to \infty \). Hence, there is no \( x \in \ell^2(\mathbb{N}) \) such that
\[
\langle ne^{(n)}, y \rangle \to \langle x, y \rangle \quad \text{as } n \to \infty.
\]

**Remark.** It is a consequence of the uniform boundedness theorem that any weakly convergent sequence is bounded in norm, as is true of the sequence in (a) but not of the sequence in (b).