Compact Sets in Metric Spaces

Math 201A, Fall 2016

1 Sequentially compact sets

Definition 1. A metric space is sequentially compact if every sequence has a convergent subsequence.

Definition 2. A metric space is complete if every Cauchy sequence converges.

Definition 3. Let $\epsilon > 0$. A set $\{x_\alpha \in X : \alpha \in I\}$ is an ϵ-net for a metric space X if

$$X = \bigcup_{\alpha \in I} B_\epsilon(x_\alpha).$$

Definition 4. A metric space is totally bounded if it has a finite ϵ-net for every $\epsilon > 0$.

Theorem 5. A metric space is sequentially compact if and only if it is complete and totally bounded.

Proof. Suppose that X is a sequentially compact metric space. Then every Cauchy sequence in X has a convergent subsequence, so, by Lemma 6 below, the Cauchy sequence converges, meaning that X is complete.

Next, suppose that X is not totally bounded. Then there exists $\epsilon_0 > 0$ such that X has no finite ϵ_0-net. Choose a sequence (x_n) in X as follows. Pick any $x_1 \in X$. Since $X \neq B_{\epsilon_0}(x_1)$, there exists $x_2 \in X$ such that $d(x_1, x_2) \geq \epsilon_0$. Given $\{x_1, x_2, \dots, x_n\}$, there exists $x_{n+1} \in X$ such that $d(x_k, x_{n+1}) \geq \epsilon_0$ for $1 \leq k \leq n$, since $X \neq \bigcup_{k=1}^n B_{\epsilon_0}(x_k)$. Then $d(x_m, x_n) \geq \epsilon_0$ for every $m \neq n$, so (x_n) has no Cauchy subsequences and therefore no convergent subsequences, meaning that X is not sequentially compact. It follows that a sequentially compact metric space is complete and totally bounded.

Conversely, suppose that a metric space X is totally bounded and complete. Let (x_n) be a sequence in X. Since X is totally bounded, it has a finite 1-net $\{a_m : 1 \leq m \leq M\}$ such that

$$X = \bigcup_{m=1}^{M} B_1(a_m).$$

At least one ball, say $X_1 = B_1(a_m)$, must contain infinitely many terms in the sequence, meaning that $x_n \in X_1$ for infinitely many $n \in \mathbb{N}$. Choose $x_{n_1} \in X_1$. A subspace of a totally bounded metric space is totally bounded with respect to its subspace metric, so X_1 has a finite $(1/2)$-net, again denoted by $\{a_m : 1 \leq m \leq M\}$, such that

$$X_1 = \bigcup_{m=1}^{M} B_{1/2}(a_m).$$

At least one ball, say $X_2 = B_{1/2}(a_m)$, contains infinitely many terms in the sequence, so we can choose $x_{n_2} \in X_2$ with $n_2 > n_1$. Continuing in this way, given $x_{n_j} \in X_j$ for $1 \leq j \leq k$ and a ball X_k of radius $1/k$ that contains infinitely many terms in the sequence, we cover X_k by finitely many balls of radius $1/(k+1)$, let X_{k+1} be one of these balls that contains infinitely many terms in the sequence, and choose $x_{n_{k+1}} \in X_{k+1}$ such that $n_{k+1} > n_k$.

Since $x_{n_j} \in X_k$ for all $j \geq k$, it follows that

$$d\left(x_{n_i}, x_{n_j}\right) < \frac{2}{k} \qquad \text{for all } i, j \geq k,$$

so the subsequence $(x_{n_k})_{k=1}^{\infty}$ is a Cauchy sequence. Since X is complete, the subsequence converges, which proves that a complete, totally bounded metric space is sequentially compact. □

Lemma 6. If a Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges to the limit of the subsequence.

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space X with a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to $x \in X$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon/2$ for all $m, n > N$, and $k \in \mathbb{N}$ such that $n_k > N$ and $d(x_{n_k}, x) < \epsilon/2$. Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon$ for all $n > N$, so $x_n \to x$ as $n \to \infty$. □

Definition 7. A metric space has the finite intersection property for closed sets if every decreasing sequence of closed, nonempty sets has nonempty intersection.

Theorem 8. A metric space is sequentially compact if and only if it has the finite intersection property for closed sets.

Proof. Suppose that X is sequentially compact. Given a decreasing sequence of closed sets F_n, choose $x_n \in F_n$ for each $n \in \mathbb{N}$. Then (x_n) has a convergent subsequence (x_{n_k}) with $x_{n_k} \to x$ as $k \to \infty$. Since $x_{n_k} \in F_n$ for all $n_k \geq n$ and F_n is closed, $x \in F_n$ for every $n \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} F_n$, and $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Conversely, suppose that X has the finite intersection property. Let (x_n) be a sequence in X and define

$$F_n = \overline{T_n}, \qquad T_n = \{x_k : k > n\}.$$

Then (F_n) is a decreasing sequence of non-empty, closed sets, so there exists

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

Choose a subsequence (x_{n_k}) of (x_n) as follows. For $k = 1$, there exists $x_{n_1} \in T_1$ such that $d(x_{n_1}, x) < 1$, since $x \in F_1$ and T_1 is dense in F_1. Similarly, since $x \in F_{n_1}$ and T_{n_1} is dense in F_{n_1}, there exists $x_{n_2} \in T_{n_1}$ with $n_2 > n_1$ such that $d(x_{n_2}, x) < 1/2$. Continuing in this way (or by induction), given x_{n_k} we choose $x_{n_{k+1}} \in T_{n_k}$, where $n_{k+1} > n_k$, such that $d(x_{n_{k+1}}, x) < 1/(k+1)$. Then $x_{n_k} \to x$ as $k \to \infty$, so X is sequentially compact. □

Lemma 9 (Lebesgue Covering Lemma)**.** If $\{G_\alpha \subset X : \alpha \in I\}$ is an open cover of a sequentially compact metric space X, then there exists $\delta > 0$ such that for every $x \in X$ there is some $\alpha \in I$ with $B_\delta(x) \subset G_\alpha$.

Proof. We use proof by contradiction. Suppose that no such $\delta > 0$ exists. Then for every $n \in \mathbb{N}$ there exists $x_n \in X$ such that $B_{1/n}(x_n)$ is not contained in G_α for any $\alpha \in I$. Since X is sequentially compact, the sequence (x_n) has a convergent subsequence (x_{n_k}). Let $x = \lim_{k\to\infty} x_{n_k}$. Then $x \in G_\alpha$ for some $\alpha \in I$ since $\{G_\alpha : \alpha \in I\}$ covers X, and $B_\epsilon(x) \subset G_\alpha$ for some $\epsilon > 0$ since G_α is open. Since $x_{n_k} \to x$ and $1/n_k \to 0$ as $k \to \infty$, there exists $k \in \mathbb{N}$ such that $d(x, x_{n_k}) < \epsilon/2$ and $1/n_k < \epsilon/2$. It follows that $B_{1/n_k}(x_{n_k}) \subset B_\epsilon(x)$, so $B_{1/n_k}(x_{n_k})$ is contained in G_α, contradicting the choice of x_{n_k}. □

2 Compact sets

Definition 10. An open cover of a metric (or topological) space X is a collection of open sets $\mathcal{C} = \{G_\alpha \subset X : \alpha \in I\}$ such that

$$X = \bigcup_{\alpha \in I} G_\alpha.$$

In that case, we say that $\mathcal{C}$ covers X. A subcover of $\mathcal{C}$ is a subcollection $\{G_\beta : \beta \in J\}$ of sets in $\mathcal{C}$, where $J \subset I$, that covers X. The subcover is finite if J is finite.

Definition 11. A metric (or topological) space is compact if every open cover of the space has a finite subcover.

Theorem 12. A metric space is compact if and only if it is sequentially compact.

Proof. Suppose that X is compact. Let (F_n) be a decreasing sequence of closed nonempty subsets of X, and let $G_n = F_n^c$.

If $\bigcup_{n=1}^{\infty} G_n = X$, then $\{G_n : n \in \mathbb{N}\}$ is an open cover of X, so it has a finite subcover $\{G_{n_k} : k = 1, 2, \dots K\}$ since X is compact. Let

$$N = \max\{n_k : k = 1, 2, \dots K\}.$$

Then $\bigcup_{n=1}^{N} G_n = X$, so

$$F_N = \bigcap_{n=1}^{N} F_n = \left(\bigcup_{n=1}^{N} G_n\right)^c = \emptyset,$$

contrary to our assumption that every F_n is nonempty.

It follows that $\bigcup_{n=1}^{\infty} G_n \neq X$ and then

$$\bigcap_{n=1}^{\infty} F_n = \left(\bigcup_{n=1}^{\infty} G_n\right)^c \neq \emptyset,$$

meaning that X has the finite intersection property for closed sets, so X is sequentially compact.

Conversely, suppose that X is sequentially compact. Let

$$\{G_\alpha \subset X : \alpha \in I\}$$

be an open cover of X. By Lemma 9, there exists $\delta > 0$ such that every ball $B_\delta(x)$ is contained in some G_α. Since X is sequentially compact, it is totally bounded, so there exists a finite collection of balls of radius δ

$$\{B_\delta(x_i) : i = 1, 2, \ldots, n\}$$

that covers X. Choose $\alpha_i \in I$ such that $B_\delta(x_i) \subset G_{\alpha_i}$. Then

$$\{G_{\alpha_i} : i = 1, 2, \ldots, n\}$$

is a finite subcover of X, so X is compact. □