Continuous Functions on Metric Spaces

Math 201A, Fall 2016

1 Continuous functions

Definition 1. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $a \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x, a) < \delta$ implies that $d_Y(f(x), f(a)) < \epsilon$.

In terms of open balls, the definition says that $f(B_\delta(a)) \subset B_\epsilon(f(a))$. In future, X, Y will denote metric spaces, and we will not distinguish explicitly between the metrics on different spaces.

Definition 2. A subset $U \subset X$ is a neighborhood of a point $a \in X$ if $B_\epsilon(a) \subset U$ for some $\epsilon > 0$.

A set is open if and only if it is a neighborhood of every point in the set, and U is a neighborhood of x if and only if $U \supset G$ where G is an open neighborhood of x. Note that slightly different definitions of a neighborhood are in use; some definitions require that a neighborhood is an open set, which we do not assume.

Proposition 3. A function $f : X \to Y$ is continuous at $a \in X$ if and only if for every neighborhood $V \subset Y$ of $f(a)$ the inverse image $f^{-1}(V) \subset X$ is a neighborhood of a.

Proof. Suppose that the condition holds. If $\epsilon > 0$, then $V = B_\epsilon(f(a))$ is a neighborhood of $f(a)$, so $U = f^{-1}(V)$ is a neighborhood of a. Then $B_\delta(a) \subset U$ for some $\delta > 0$, which implies that $f(B_\delta(a)) \subset B_\epsilon(f(a))$, so f is continuous at a.

Conversely, if f is continuous at a and V is a neighborhood of $f(a)$, then $B_\epsilon(f(a)) \subset V$ for some $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that $f(B_\delta(a)) \subset B_\epsilon(f(a))$, so $B_\delta(a) \subset f^{-1}(V)$, meaning that $f^{-1}(V)$ is a neighborhood of a. $\square$

Definition 4. A function $f : X \to Y$ is sequentially continuous at $a \in X$ if $x_n \to a$ in X implies that $f(x_n) \to f(a)$ in Y.

Theorem 5. A function $f : X \to Y$ is continuous at a if and only if it is sequentially continuous at a.

Proof. Suppose that f is continuous at a. Let $\epsilon > 0$ be given and suppose that $x_n \to a$. Then there exists $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ for $d(x, a) < \delta$, and there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \delta$ for $n > N$. It follows that $d(f(x_n), f(a)) < \epsilon$ for $n > N$, so $f(x_n) \to f(a)$ and f is sequentially continuous at a.

Conversely, suppose that f is not continuous at a. Then there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in X$ with $d(x_n, a) < 1/n$ and $d(f(x_n), f(a)) \geq \epsilon_0$. Then $x_n \to a$ but $f(x_n) \not\to f(a)$, so f is not sequentially continuous at a. □

Definition 6. A function $f : X \to Y$ is continuous if f is continuous at every $x \in X$.

Theorem 7. A function $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every V that is open in Y.

Proof. Suppose that the inverse image under f of every open set is open. If $x \in X$ and $V \subset Y$ is a neighborhood of $f(x)$, then $V \supset W$ where W is an open neighborhood of $f(x)$. Then $f^{-1}(W)$ is an open neighborhood of x and $f^{-1}(W) \subset f^{-1}(V)$, so $f^{-1}(V)$ is a neighborhood of x, which shows that f is continuous.

Conversely, suppose that $f : X \to Y$ is continuous and $V \subset Y$ is open. If $x \in f^{-1}(V)$, then V is an open neighborhood of $f(x)$, so the continuity of f implies that $f^{-1}(V)$ is a neighborhood of x. It follows that $f^{-1}(V)$ is open since it is a neighborhood of every point in the set. □

Theorem 8. The composition of continuous functions is continuous

Proof. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous, and $g \circ f : X \to Z$ is their composition. If $W \subset Z$ is open, then $V = g^{-1}(W)$ is open, so $U = f^{-1}(V)$ is open. It follows that $(g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right)$ is open, so $g \circ f$ is continuous. □

Theorem 9. The continuous image of a compact set is compact.

Proof. Suppose that $f : X \to Y$ is continuous and X is compact. If $\{G_\alpha : \alpha \in I\}$ is an open cover of $f(X)$, then $\{f^{-1}(G_\alpha) : \alpha \in I\}$ is an open cover of X, since the inverse image of an open set is open. Since X is compact, it has a finite subcover $\{f^{-1}(G_{\alpha_i}) : i = 1, 2, \dots, n\}$. Then $\{G_{\alpha_i} : i = 1, 2, \dots, n\}$ is a finite subcover of $f(X)$, which proves that $f(X)$ is compact. □

2 Uniform convergence

A subset $A \subset X$ is bounded if $A \subset B_R(x)$ for some (and therefore every) $x \in X$ and some $R > 0$. Equivalently, A is bounded if

$$\operatorname{diam} A = \sup\{d(x, y) : x, y \in A\} < \infty.$$

Definition 10. A function $f : X \to Y$ is bounded if $f(X) \subset Y$ is bounded.

Definition 11. A sequence (f_n) of functions $f_n : X \to Y$ converges uniformly to a function $f : X \to Y$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$.

Unlike pointwise convergence, uniform convergence preserves boundedness and continuity.

Proposition 12. Let (f_n) be a sequence of functions $f_n : X \to Y$. If each f_n is bounded and $f_n \to f$ uniformly, then $f : X \to Y$ is bounded.

Proof. By the uniform convergence, there exists $n \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) \le 1 \qquad \text{for all } x \in X.$$

Since f_n is bounded, there exists $y \in Y$ and $R > 0$ such that

$$d(f_n(x), y) \le R \qquad \text{for all } x \in X.$$

It follows that

$$d(f(x), y) \le d(f(x), f_n(x)) + d(f_n(x), y) \le R + 1 \quad \text{for all } x \in X,$$

meaning that f is bounded. □

Theorem 13. Let (f_n) be a sequence of functions $f_n : X \to Y$. If each f_n is continuous at $a \in X$ and $f_n \to f$ uniformly, then $f : X \to Y$ is continuous at as.

Proof. Let $a \in X$. Since (f_n) converges uniformly to f, given $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$d\left(f_n(x), f(x)\right) < \frac{\epsilon}{3} \qquad \text{for all } x \in X,$$

and since f_n is continuous at a, there exists $\delta > 0$ such that

$$d\left(f_n(x), f_n(a)\right) < \frac{\epsilon}{3} \qquad \text{if } d(x, a) < \delta.$$

If $d(x, a) < \delta$, then it follows that

$$d\left(f(x), f(a)\right) \leq d\left(f(x), f_n(x)\right) + d\left(f_n(x), f_n(a)\right) + d\left(f_n(a), f(a)\right) < \epsilon,$$

which proves that f is continuous at a. □

Since uniform convergence preserves continuity at a point, the uniform limit of continuous functions is continuous.

Definition 14. A sequence (f_n) of functions $f_n : X \to Y$ is uniformly Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n > N$ implies that $d\left(f_m(x), f_n(x)\right) < \epsilon$ for all $x \in X$.

A uniformly convergent sequence of functions is uniformly Cauchy. The converse is also true for functions that take values in a complete metric space.

Theorem 15. Let Y be a complete metric space. Then a uniformly Cauchy sequence (f_n) of functions $f_n : X \to Y$ converges uniformly to a function $f : X \to Y$.

Proof. The uniform Cauchy condition implies that the sequence $(f_n(x))$ is Cauchy in Y for every $x \in X$. Since Y is complete, $f_n(x) \to f(x)$ as $n \to \infty$ for some $f(x) \in Y$.

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for all $m, n > N$ we have

$$d\left(f_m(x), f_n(x)\right) < \epsilon \qquad \text{for all } x \in X.$$

Taking the limit of this inequality as $m \to \infty$, we get that

$$d\left(f(x), f_n(x)\right) \leq \epsilon \qquad \text{for all } x \in X$$

for all $n > N$, which shows that (f_n) converges uniformly to f as $n \to \infty$. □

3 Function spaces

Definition 16. The metric space $(B(X,Y), d_\infty)$ is the space of bounded functions $f : X \to Y$ equipped with the uniform metric

$$d_\infty(f,g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

It is straightforward to check that d_∞ is a metric on $B(X,Y)$; in particular, $d_\infty(f,g) < \infty$ if f, g are bounded functions.

Theorem 17. Let Y be a complete metric space. Then $(B(X,Y), d_\infty)$ is a complete metric space.

Proof. If (f_n) is a Cauchy sequence in $B(X,Y)$, then (f_n) is uniformly Cauchy, so by Theorem 15 it converges uniformly to a function $f : X \to Y$. By Proposition 12, $f \in B(X,Y)$, so $B(X,Y)$ is complete. □

Definition 18. The space $C(X,Y)$ is the space of continuous functions $f : X \to Y$, and $(C_b(X,Y), d_\infty)$ is the space of bounded, continuous functions $f : X \to Y$ equipped with the uniform metric d_∞.

Theorem 19. Let X be a metric space and Y a complete metric space. Then $(C_b(X,Y), d_\infty)$ is a complete metric space.

Proof. By Theorem 13, $C_b(X,Y)$ is a closed subspace of the complete metric space $B(X,Y)$, so it is a complete metric space. □

4 Continuous functions on compact sets

Definition 20. A function $f : X \to Y$ is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ and $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

Theorem 21. A continuous function on a compact metric space is bounded and uniformly continuous.

Proof. If X is a compact metric space and $f : X \to Y$ a continuous function, then $f(X)$ is compact and therefore bounded, so f is bounded.

Let $\epsilon > 0$. For each $a \in X$, there exists $\delta(a,\epsilon) > 0$ such that

$$d(f(x), f(a)) < \frac{\epsilon}{2} \qquad \text{for all } x \in X \text{ with } d(x,a) < \delta(a,\epsilon).$$

Then $\left\{B_{\delta(a,\epsilon)}(a) : a \in X\right\}$ is an open cover of X, so by the Lebesgue covering lemma, there exists $\eta > 0$ such that for every $x \in X$ there exists $a \in X$ with $B_\eta(x) \subset B_{\delta(a,\epsilon)}(a)$. Hence, $d(x, y) < \eta$ implies that $x, y \in B_{\delta(a,\epsilon)}(a)$, so

$$d(f(x), f(y)) \leq d(f(x), f(a)) + d(f(a), f(y)) < \epsilon,$$

which shows that f is uniformly continuous. $\square$

For real-valued functions, we have the following basic result.

Theorem 22 (Weierstrass)**.** If X is compact and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its maximum and minimum values.

Proof. The image $f(X) \subset \mathbb{R}$ is compact, so it is closed and bounded. It follows that $M = \sup_X f < \infty$ and $M \in f(X)$. Similarly, $\inf_X f \in f(X)$. $\square$

5 The Arzelà-Ascoli theorem

Definition 23. A family of functions $\mathcal{F} \subset C(X, Y)$ is equicontinuous if for every $a \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $d(x, a) < \delta$, then $d(f(x), f(a)) < \epsilon$ for every $f \in \mathcal{F}$.

Definition 24. A family of functions $\mathcal{F} \subset C(X, Y)$ is uniformly equicontinuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$ for every $f \in \mathcal{F}$.

Theorem 25. If X is a compact metric space and $\mathcal{F} \subset C(X, Y)$ is equicontinuous, then $\mathcal{F}$ is uniformly equicontinuous.

Proof. Let $\mathcal{F} \subset C(X, Y)$ be equicontinuous, and suppose that $\epsilon > 0$. For each $a \in X$, there exists $\delta(a, \epsilon) > 0$ such that if $x \in X$ and $d(x, a) < \delta(a, \epsilon)$, then $d(f(x), f(a)) < \epsilon/2$ for every $f \in \mathcal{F}$. Then $\{B_{\delta(a,\epsilon)}(a) : a \in X\}$ is an open cover of X, so by the Lebesgue covering lemma there exist $\eta > 0$ such that for every $x \in X$ we have $B_\eta(x) \subset B_{\delta(a,\epsilon)}(a)$ for some $a \in X$. If $d(x, y) < \eta$, then $x, y \in B_{\delta(a,\epsilon)}(a)$, so

$$d(f(x), f(y)) \leq d(f(x), f(a)) + d(f(a), f(y)) < \epsilon \qquad \text{for every } f \in \mathcal{F},$$

which shows that $\mathcal{F}$ is uniformly equicontinuous. $\square$

For simplicity, we now specialize to real-valued functions. In that case, we write $C(X) = C(X, \mathbb{R})$. If X is compact, then $C(X)$ equipped with the sup-norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$, and the usual pointwise definitions of vector addition and scalar multiplication, is a Banach space.

Definition 26. A family of functions $\mathcal{F} \subset C(X)$ is pointwise bounded if for every $x \in X$ there exists $M > 0$ such that $|f(x)| \le M$ for all $f \in \mathcal{F}$.

Definition 27. A subset $A \subset X$ is of a metric space X is precompact (or relatively compact) if its closure $\bar{A}$ is compact.

Lemma 28. Let X be a complete metric space. A subset $A \subset X$ is precompact if and only if it is totally bounded.

Proof. If A is precompact, then the compact set $\bar{A}$ is totally bounded, so $A \subset \bar{A}$ is totally bounded.

Conversely, suppose that A is totally bounded. If $\epsilon > 0$, then A has a finite $(\epsilon/2)$-net E, and every $x \in \bar{A}$ belongs to $\bar{B}_{\epsilon/2}(a)$ for some $a \in E$. It follows that E is a finite ϵ-net for $\bar{A}$, so $\bar{A}$ is totally bounded. Moreover, $\bar{A}$ is a closed subset of a complete space, so $\bar{A}$ is complete, which shows that $\bar{A}$ is compact and A is precompact. $\square$

Theorem 29 (Arzelà-Ascoli)**.** Let X be a compact metric space. A family $\mathcal{F} \subset C(X)$ of continuous functions $f : X \to \mathbb{R}$ is precompact with respect to the sup-norm topology if and only if it is pointwise bounded and equicontinuous. Furthermore, $\mathcal{F}$ is compact if and only if it is closed, pointwise bounded, and equicontinuous.

Proof. Since $C(X)$ is complete, a subset is complete if and only if it is closed. It follows that $\mathcal{F}$ is compact if and only if it is closed and totally bounded. From Lemma 28, $\mathcal{F}$ is precompact if and only if it is totally bounded. Thus, it suffices to prove that $\mathcal{F}$ is totally bounded if and only if it is pointwise bounded and equicontinuous.

First, suppose that $\mathcal{F}$ is pointwise bounded and equicontinuous, and let $\epsilon > 0$. We will construct a finite ϵ-net for $\mathcal{F}$.

The family $\mathcal{F}$ is equicontinuous and X is compact, so $\mathcal{F}$ is uniformly equicontinuous, and there exists $\delta > 0$ such that

$$d(x, y) < \delta \text{ implies that} \qquad |f(x) - f(y)| < \frac{\epsilon}{3} \text{ for all } f \in \mathcal{F}.$$

Since X is compact, it has a finite δ-net $E = \{x_i : 1 \leq i \leq n\}$ such that

$$X = \bigcup_{i=1}^{n} B_\delta(x_i).$$

Furthermore, since $\mathcal{F}$ is pointwise bounded,

$$M_i = \sup\{|f(x_i)| : f \in \mathcal{F}\} < \infty \qquad \text{for each } 1 \leq i \leq n.$$

Let $M = \max\{M_i : 1 \leq i \leq n\}$. Then $[-M, M] \subset \mathbb{R}$ is compact, so it has a finite $(\epsilon/6)$-net $F = \{y_j \in \mathbb{R} : 1 \leq j \leq m\}$ such that

$$[-M, M] \subset \bigcup_{j=1}^{m} B_{\epsilon/6}(y_j).$$

Here, $B_{\epsilon/6}(y_j)$ is an open interval in $\mathbb{R}$ of diameter $\epsilon/3$.

Let Φ be the finite set of maps $\phi : E \to F$. For each $\phi \in \Phi$, define

$$\mathcal{F}_\phi = \left\{ f \in \mathcal{F} : f(x_i) \in B_{\epsilon/6}\left(\phi(x_i)\right) \text{ for } i = 1, \dots, n \right\}.$$

Since $f(x_i) \in [-M, M]$ and $\{B_{\epsilon/6}(y_j) : 1 \leq j \leq m\}$ covers $[-M, M]$, we have

$$\mathcal{F} = \bigcup_{\phi \in \Phi} \mathcal{F}_\phi.$$

Let $f, g \in \mathcal{F}_\phi$. Then for each $x_i \in E$, we have $f(x_i), g(x_i) \in B_{\epsilon/6}(y_j)$ for some $y_j \in F$, so

$$|f(x_i) - g(x_i)| < \frac{\epsilon}{3}.$$

Furthermore, if $x \in X$, then $x \in B_\delta(x_i)$ for some $x_i \in E$, so $d(x, x_i) < \delta$, and the uniform equicontinuity of $\mathcal{F}$ implies that

$$|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| < \epsilon.$$

Thus, $\|f - g\|_\infty < \epsilon$, so the diameter of $\mathcal{F}_\phi$ is less than ϵ, and $\mathcal{F}_\phi \subset B_\epsilon(f_\phi)$ for some $f_\phi \in \mathcal{F}$. Hence,

$$\mathcal{F} \subset \bigcup_{\phi \in \Phi} B_\epsilon(f_\phi)$$

is totally bounded.

Conversely, suppose that $\mathcal{F} \subset C(X)$ is precompact. Then $\mathcal{F}$ is bounded, so there exists $M > 0$ such that $\|f\|_\infty \leq M$ for all $f \in \mathcal{F}$, which implies that $\mathcal{F}$ is pointwise (and, in fact, uniformly) bounded.

Moreover, $\mathcal{F}$ is totally bounded, so given $\epsilon > 0$, there exists a finite $(\epsilon/3)$-net $E = \{f_i : 1 \leq i \leq n\}$ for $\mathcal{F}$. Each $f_i : X \to \mathbb{R}$ is uniformly continuous, so there exists $\delta_i > 0$ such that $d(x, y) < \delta_i$ implies that $|f_i(x) - f_i(y)| < \epsilon/3$. Let $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. If $f \in \mathcal{F}$, then $\|f - f_i\|_\infty < \epsilon/3$ for some $f_i \in E$, so $d(x, y) < \delta$ implies that

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \epsilon,$$

meaning that $\mathcal{F}$ is uniformly equicontinuous. □

This result, and the proof, also applies to complex-valued functions in $C(X, \mathbb{C})$, vector-valued functions in $C(X, \mathbb{R}^n)$, and — with a stronger hypothesis — to functions that take values in a complete metric space Y: If X is compact, then $\mathcal{F} \subset C(X, Y)$ is precompact with respect to the uniform metric d_∞ if and only if $\mathcal{F}$ is equicontinuous and pointwise precompact, meaning that the set $\{f(x) : f \in \mathcal{F}\}$ is precompact in Y for every $x \in X$.

6 The Peano existence theorem

As an application of the Arzelà-Ascoli theorem, we prove an existence result for an initial-value problem for a scalar ODE. (Essentially the same proof applies to systems of ODEs.)

Theorem 30. Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $u_0 \in \mathbb{R}$. Then there exists $T > 0$ such that the initial-value problem

$$\begin{aligned} \frac{du}{dt} &= f(u, t), \\ u(0) &= u_0 \end{aligned}$$

has a continuously differentiable solution $u : [0, T] \to \mathbb{R}$.

Proof. For each $h > 0$, construct an approximate solution $u_h : [0, \infty) \to \mathbb{R}$ as follows. Let $t_n = nh$ and define a sequence $(u_n)_{n=0}^{\infty}$ recursively by

$$u_{n+1} = u_n + hf(u_n, t_n).$$

Define $u_h(t)$ by linear interpolation between u_n and u_{n+1}, meaning that

$$u_h(t) = u_n + \frac{(t - t_n)}{h}(u_{n+1} - u_n) \qquad \text{for } t_n \leq t \leq t_{n+1}.$$

Choose $L, T_1 > 0$, and let

$$R_1 = \left\{(x, t) \in \mathbb{R}^2 : |x - u_0| \leq L \text{ and } 0 \leq t \leq T_1\right\}.$$

Then, since f is continuous and R_1 is compact,

$$M = \sup_{(x,t)\in R_1} |f(x, t)| < \infty.$$

If $(u_k, t_k) \in R_1$ for every $0 \leq k \leq n - 1$, then

$$|u_n - u_0| \leq \sum_{k=0}^{n-1} |u_{k+1} - u_k| = h \sum_{k=0}^{n-1} |f(u_k, t_k)| \leq M t_n.$$

Thus, $(u_n, t_n) \in R_1$ so long as $Mt_n \leq L$ and $t_n \leq T_1$. Let

$$T = \min\left\{T_1, \frac{L}{M}\right\}, \qquad N = \frac{T}{h},$$

and define $R \subset R_1$ by

$$R = \left\{(x, t) \in \mathbb{R}^2 : |x - u_0| \leq L \text{ and } 0 \leq t \leq T\right\}.$$

Then it follows that $(u_n, t_n) \in R$ if $0 \leq t_n \leq T$ and $0 \leq n \leq N$. Moreover,

$$|u_{n+1} - u_n| = h|f(u_n, t_n)| \leq Mh \qquad \text{for every } 0 \leq n \leq N.$$

It is clear that the linear interpolant u_h then satisfies

$$|u_h(s) - u_h(t)| \leq M|s - t| \quad \text{for every } 0 \leq s, t \leq T.$$

To show this explicitly, suppose that $0 \leq s < t \leq T$, $t_m \leq s \leq t_{m+1}$, and

$t_n \leq t \leq t_{n+1}$. Then

$$\begin{aligned}
|u_h(t) - u_h(s)| &\leq |u_h(t) - u_h(t_n)| + \sum_{k=m+1}^{n-1} |u_h(t_{k+1}) - u_h(t_k)| \\
&\qquad\qquad + |u_h(t_{m+1}) - u_h(s)| \\
&\leq (t - t_n)\, |f(u_n, t_n)| + h \sum_{k=m+1}^{n-1} |f(u_k, t_k)| \\
&\qquad\qquad + (t_{m+1} - s)\, |f(u_m, t_m)| \\
&\leq M\left[(t - t_n) + (t_n - t_{m+1}) + (t_{m+1} - s)\right] \\
&\leq M|t - s|.
\end{aligned}$$

In addition, $|u_h(t)| \leq L$ for $t \in [0, T]$.

By the Arzelà-Ascoli theorem, the family $\{u_h : h > 0\}$ is precompact in $C([0, T])$, so we can extract a subsequence $(u_{h_j})_{j=1}^{\infty}$ such that $h_j \to 0$ and $u_{h_j} \to u$ uniformly to some $u \in C([0, T])$ as $j \to \infty$.

To show that u is a solution of the initial-value problem, we note that, by the fundamental theorem of calculus, $u \in C^1([0, T])$ is a solution if and only if $u \in C([0, T])$ and u satisfies the integral equation

$$u(t) = u_0 + \int_0^t f\left(u(s), s\right)\, ds \qquad \text{for } 0 \leq t \leq T.$$

Let $\chi_{[0,t]}$ be the characteristic function of the interval $[0, t]$, defined by

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq t \\ 0 & \text{if } s > t \end{cases}$$

Then the piecewise-linear approximation u_h satisfies the equation

$$u_h(t) = u_0 + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \chi_{[0,t]}(s) f\left(u_h(t_k), t_k\right)\, ds.$$

Let $\epsilon > 0$. Since $f : R \to \mathbb{R}$ is continuous on a compact set, it is uniformly continuous, and there exists $\delta > 0$ such that

$$|x - y| < M\delta \text{ and } |s - t| < \delta \text{ implies that } |f(x, s) - f(y, t)| < \epsilon/T.$$

For $0 < h < \delta$ and $t_k \leq s \leq t_{k+1}$, we have $|u_h(s) - u_h(t_k)| < M\delta$ and $|s - t_k| < \delta$, so for $0 \leq t \leq T$ we get that

$$\begin{aligned}
&\left|u_h(t) - u_0 - \int_0^t f\left(u_h(s), s\right)\, ds\right| \\
&\qquad \leq \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \chi_{[0,t]}(s)\left|f((u_h(s), s) - f\left(u_h(t_k), t_k\right)\right|\, ds \\
&\qquad < \frac{\epsilon}{T} \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \chi_{[0,t]}(s)\, ds \\
&\qquad < \epsilon.
\end{aligned}$$

Thus,

$$u_h(t) - u_0 - \int_0^t f\left(u_h(s), s\right)\, ds \to 0 \qquad \text{uniformly on } [0, T] \text{ as } h \to 0.$$

Since $u_{h_j} \to u$ uniformly as $j \to \infty$, it follows that

$$u_0 + \int_0^t f\left(u_{h_j}(s), s\right)\, ds \to u(t) \qquad \text{uniformly on } [0, T] \text{ as } j \to \infty.$$

However, the uniform convergence $u_{h_j} \to u$ and the uniform continuity of f imply that $f(u_{h_j}(t), t) \to f(u(t), t)$ uniformly. Then, since the Riemann integral of the limit of a uniformly convergent sequence of Riemann-integrable functions is the limit of the Riemann integrals, we get that

$$u_0 + \int_0^t f\left(u_{h_j}(s), s\right)\, ds \to u_0 + \int_0^t f\left(u(s), s\right)\, ds \qquad \text{as } j \to \infty$$

for each $t \in [0, T]$. Hence, since the limits must be the same,

$$u(t) = u_0 + \int_0^t f\left(u(s), s\right)\, ds,$$

so $u \in C^1([0, T])$ is a solution of the initial-value problem. $\square$

7 The Stone-Weierstrass theorem

Definition 31. A subset $A \subset X$ is dense in X if $\bar{A} = X$.

Definition 32. A family of functions $\mathcal{A} \subset C(X)$ is an algebra if it is a linear subspace of $C(X)$ and $f, g \in \mathcal{A}$ implies that $fg \in \mathcal{A}$.

Here, fg is the usual pointwise product, $(fg)(x) = f(x)g(x)$.

Definition 33. A family of functions $\mathcal{F} \subset C(X)$ separates points if for every pair of distinct points $x, y \in X$ there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

A constant function $f : X \to \mathbb{R}$ is a function such that $f(x) = c$ for all $x \in X$ and some $c \in \mathbb{R}$.

Theorem 34 (Stone-Weierstrass)**.** Let X be a compact metric space. If $\mathcal{A} \subset C(X)$ is an algebra that separates points and contains the constant functions, then $\mathcal{A}$ is dense in $(C(X), \|\cdot\|_\infty)$.

Proof. Suppose that $f \in C(X)$ and let $\epsilon > 0$. By Lemma 35, for every pair of points $y, z \in X$, there exists $g_{yz} \in \mathcal{A}$ such that $g_{yz}(y) = f(y)$ and $g_{yz}(z) = f(z)$.

First, fix $y \in X$. Since $f - g_{yz}$ is continuous and $(f - g_{yz})(z) = 0$, for each $z \in X$ there exists $\delta(z) > 0$ such that

$$g_{yz}(x) < f(x) + \epsilon \qquad \text{for all } x \in B_{\delta(z)}(z).$$

The family of open balls $\{B_{\delta(z)}(z) : z \in X\}$ covers X, so it has a finite subcover $\{B_{\delta_i}(z_i) : 1 \leq i \leq n\}$ since X is compact. By Lemma 37, the function

$$g_y = \min\{g_{yz_i} : 1 \leq i \leq n\}$$

belongs to $\bar{\mathcal{A}}$. Moreover, $g_y(y) = f(y)$ and

$$g_y(x) < f(x) + \epsilon \qquad \text{for all } x \in X.$$

Next, consider g_y. Since $f - g_y$ is continuous and $(f - g_y)(y) = 0$, for each $y \in Y$ there exists $\delta(y) > 0$ such that

$$g_y(x) > f(x) - \epsilon \qquad \text{for all } x \in B_{\delta(y)}(y).$$

Then $\{B_{\delta(y)}(y) : y \in X\}$ is an open cover of X, so it has a finite subcover $\{B_{\delta(y_j)}(y_j) : 1 \leq j \leq m\}$. By Lemma 37, the function

$$g = \max\{g_{y_j} : 1 \leq j \leq m\}$$

belongs to $\bar{\mathcal{A}}$. Furthermore,

$$f(x) - \epsilon < g(x) < f(x) + \epsilon \qquad \text{for all } x \in X,$$

so $\|f - g\|_\infty < \epsilon$, which shows that $\bar{\mathcal{A}}$ is dense in $C(X)$. $\square$

To complete the proof of the Stone-Weierstrass theorem, we prove several lemmas.

Lemma 35. Let $\mathcal{A} \subset C(X)$ be a linear subspace that contains the constant functions and separates points. If $f \in C(X)$ and $y, z \in X$, then there exists $g_{yz} \in \mathcal{A}$ such that $g_{yz}(y) = f(y)$ and $g_{yz}(z) = f(z)$.

Proof. Given distinct points $y, z \in X$ and real numbers $a, b \in \mathbb{R}$, there exists $h \in \mathcal{A}$ with $h(y) \neq h(z)$. Then $g \in \mathcal{A}$ defined by

$$g(x) = a + (b - a) \left[\frac{h(x) - h(y)}{h(z) - h(y)} \right]$$

satisfies $g(y) = a$, $g(z) = b$. If $y \neq z$, choose g_{yz} as above with $a = f(y)$ and $b = f(z)$; if $y = z$, choose $g_{yy} = f$. $\square$

The next lemma is a special case of the Weierstrass approximation theorem.

Lemma 36. There is a sequence (p_n) of polynomials $p_n : [-1, 1] \to \mathbb{R}$ such that $p_n(t) \to |t|$ as $n \to \infty$ uniformly on $[-1, 1]$.

Proof. Define a sequence of polynomials $q_n : [0, 1] \to \mathbb{R}$ by the recursion relation

$$q_{n+1}(t) = q_n(t) + \frac{1}{2} \left(t - q_n^2(t) \right), \qquad q_0(t) = 0.$$

We claim that for every $n \in \mathbb{N}$, we have

$$q_{n-1}(t) \leq q_n(t) \leq \sqrt{t} \qquad \text{for all } t \in [0, 1].$$

Suppose as an induction hypothesis that this inequality holds for some $n \in \mathbb{N}$. Then $q_{n+1}(t) \geq q_n(t)$ and

$$\begin{aligned} \sqrt{t} - q_{n+1}(t) &= \sqrt{t} - q_n(t) - \frac{1}{2} \left(t - q_n^2(t) \right) \\ &= \left(\sqrt{t} - q_n(t) \right) \left(1 - \frac{1}{2} \left(\sqrt{t} + q_n(t) \right) \right) \\ &\geq 0. \end{aligned}$$

so $q_n(t) \leq q_{n+1}(t) \leq \sqrt{t}$. Moreover, $q_1(t) = t/2$, so $q_0(t) \leq q_1(t) \leq \sqrt{t}$, and the inequality follows by induction.

For each $t \in [0,1]$ the real sequence $(q_n(t))$ is monotone increasing and bounded from above by 1, so it converges to some limit $q(t)$ as $n \to \infty$. Taking the limit of the recursion relations as $n \to \infty$, we find that $q^2(t) = t$, and since $q(t) \geq 0$, we have $q(t) = \sqrt{t}$.

According to Dini's theorem, a monotone increasing sequence of continuous functions that converges pointwise on a compact set to a continuous function converges uniformly, so $q_n(t) \to \sqrt{t}$ uniformly on $[0,1]$.

Finally, defining the polynomials $p_n : [-1,1] \to \mathbb{R}$ by $p_n(t) = q_n(t^2)$, we see that $p_n(t) \to |t|$ uniformly on $[-1,1]$. □

Lemma 37. Let X be a compact metric space, and suppose that $\mathcal{A} \subset C(X)$ is an algebra. If $f, g \in \mathcal{A}$, then $\max\{f,g\}, \min\{f,g\} \in \bar{\mathcal{A}}$.

Proof. Since

$$\max\{f,g\} = \frac{1}{2}\left(f + g + |f-g|\right), \qquad \min\{f,g\} = \frac{1}{2}\left(f + g - |f-g|\right)$$

it is suffices to show that $|f| \in \bar{\mathcal{A}}$ for every $f \in \mathcal{A}$.

If $f \in \mathcal{A}$, then f is bounded, so $f(X) \subset [-M, M]$ for some $M > 0$. From Lemma 36, there exists a sequence (p_n) of polynomials that converges uniformly to $|t|$ on $[-1,1]$. Then $p_n(f/M) \in \mathcal{A}$ and $p_n(f/M) \to |f|/M$ uniformly as $n \to \infty$, since p_n is uniformly continuous on $[-1,1]$. It follows that $|f| \in \bar{\mathcal{A}}$. □

Definition 38. A metric space is separable if it has a countable dense subset.

Corollary 39. Let $X \subset \mathbb{R}^n$ be a closed, bounded set and $\mathcal{P}$ the set of polynomials $p : X \to \mathbb{R}$. Then $\mathcal{P}$ is dense in $(C(X), \|\cdot\|_\infty)$. Moreover, $C(X)$ is separable.

Proof. A closed, bounded set in $\mathbb{R}^n$ is compact, and the set $\mathcal{P}$ of polynomials is a subalgebra of $C(X)$ that contains the constant functions and separates points, so $\mathcal{P}$ is dense in $C(X)$. Any polynomial can be uniformly approximated on a bounded set by a polynomial with rational coefficients, and there are countable many such polynomials, so $C(X)$ is separable. □

In particular, we have the following special case.

Theorem 40 (Weierstrass approximation)**.** If $f : [0,1] \to \mathbb{R}$ is a continuous function and $\epsilon > 0$, then there exists a polynomial $p : [0,1] \to \mathbb{R}$ such that $|f(x) - p(x)| < \epsilon$ for every $x \in [0,1]$.