Problem 1. Let $(X,d)$ be a metric space.
(a) Prove the reverse triangle inequality: for every $x, y, z \in X$
\[ d(x, y) \geq |d(x, z) - d(z, y)|. \]
(b) Prove that if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$.

Solution

• (a) The triangle inequality
\[ d(x, y) + d(y, z) \geq d(x, z) \]
implies that
\[ d(x, y) \geq d(x, z) - d(y, z). \]
Exchanging $x$ and $y$, and using the symmetry of $d$, we also have
\[ d(x, y) \geq d(y, z) - d(x, z). \]
Hence
\[ d(x, y) \geq |d(x, z) - d(y, z)|. \]

• (b) Using the reverse triangle inequality, we get that
\[
|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x, y_n)| + |d(x, y_n) - d(x, y)| \\
\leq d(x_n, x) + d(y_n, y) \\
\to 0 \quad \text{as } n \to \infty.
\]
**Problem 2.** Let $E$ be a finite set and let $P = \mathcal{P}(E)$ be the power set of $E$ (the set of all subsets of $E$). Define $d : P \times P \to \mathbb{R}$ by

$$d(A, B) = \text{card}(A \triangle B)$$

where $\text{card}(A)$ is the number of elements of $A$ and

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

is the symmetric difference of $A, B \subset E$. Show that $(P, d)$ is a metric space.

**Solution**

- We have $d(A, B) \geq 0$. If $d(A, B) = 0$, then $A \setminus B = A \cap B^c = \emptyset$, so $B \supset A$. Similarly, $A \supset B$, so $A = B$.

- The symmetry of $d$ is immediate.

- Let $A, B, C \subset X$. Then

$$A \triangle B = (A \cap B^c) \cup (A^c \cap B)$$

$$= (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B \cap C^c)$$

$$= F \cup G,$$

where (draw a Venn diagram!)

$$F = (A^c \cap B \cap C) \cup (A \cap B^c \cap C^c),$$

$$G = (A \cap B^c \cap C) \cup (A^c \cap B \cap C^c).$$

- If $x \in F$, then either $x \in A^c \cap B$ and $x \in C$, which implies that $x \notin G$, or $x \in A \cap B^c$ and $x \in C^c$, which also implies that $x \notin G$. It follows that $F \cap G = \emptyset$ and

$$\text{card}(A \triangle B) = \text{card}(F) + \text{card}(G).$$

- We have

$$F \subset (A^c \cap C) \cup (A \cap C^c) = A \Delta C,$$

so $\text{card}(F) \leq \text{card}(A \Delta C)$. Similarly, $\text{card}(G) \leq \text{card}(B \Delta C)$, which shows that

$$\text{card}(A \triangle B) \leq \text{card}(A \Delta C) + \text{card}(B \Delta C).$$

Thus, $d$ satisfies the triangle inequality.

**Remark.** In coding theory, $d$ is called the Hamming metric, which measures the number of mismatches between two finite strings of 0s and 1s.
Problem 3. If $(X, d)$ is a metric space, define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$ 

(a) Show that $(X, \rho)$ is a metric space.
(b) Show that $(X, d)$ and $(X, \rho)$ have the same convergent sequences and the same metric topologies. Do they necessarily have the same Cauchy sequences?

Solution

- (a) Let $s, t \geq 0$. Then

$$\frac{s + t}{1 + s + t} = \frac{s}{1 + s + t} + \frac{t}{1 + s + t} \leq \frac{s}{1 + s} + \frac{t}{1 + t}.$$ 

Moreover,

$$\frac{s - t}{1 + s} - \frac{t}{1 + t} = \frac{s - t}{(1 + s)(1 + t)},$$

so $0 \leq t \leq s$ implies that

$$\frac{t}{1 + t} \leq \frac{s}{1 + s}.$$

- The positivity and symmetry of $\rho$ are immediate.

- Let $x, y, z \in X$. Using the triangle inequality for $d$ and the previous inequalities, we get that

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(y, z)}{1 + d(x, z) + d(y, z)} \leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(y, z)} \leq \rho(x, z) + \rho(y, z),$$

so $\rho$ satisfies the triangle inequality, and $(X, \rho)$ is a metric space.
• (b) Clearly, \(d(x_n, x) \to 0\) if and only if \(\rho(x_n, x) \to 0\), so \(d\) and \(\rho\) have the same convergent sequences.

• Let \(B_r(x)\) denote the open ball with respect to \(d\) and \(C_r(x)\) the open ball with respect to \(\rho\). If \(d(x, y) < r\), then \(\rho(x, y) < r\), so \(B_r(x) \subset C_r(x)\). It follows that if \(G\) is open with respect to \(\rho\) and \(C_\epsilon(x) \subset G\) for each \(x \in G\) and some \(\epsilon > 0\), then \(B_\epsilon(x) \subset G\), so \(G\) is open with respect to \(d\).

• Similarly, if \(\rho(x, y) < r\) where \(r < 1/2\), then \(d(x, y) < 2r\), so \(C_r(x) \subset B_{2r}(x)\). If \(G\) is open with respect to \(d\) and \(B_\epsilon(x) \subset G\), then we can choose \(\epsilon < 1/2\) without loss of generality, and \(C_{\epsilon/2}(x) \subset G\), so \(G\) is open with respect to \(\rho\).

• The two metrics have the same Cauchy sequences. Suppose that \((x_n)\) is Cauchy in \((X, \rho)\) and let \(\epsilon > 0\). Choose \(N \in \mathbb{N}\) such that

\[
\rho(x_m, x_n) < \min \left\{ \frac{\epsilon}{2}, \frac{1}{2} \right\} \quad \text{for all } m, n > N.
\]

Then \(d(x_m, x_n) < \epsilon\) for all \(m, n > N\), so \((x_n)\) is Cauchy in \((X, d)\). The converse is similar.
Problem 4. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x, y) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}, \quad x = (x_1, x_2), \ y = (y_1, y_2).$$

(a) Show that $(\mathbb{R}^2, d)$ is a metric space. Is this metric derived from a norm $\| \cdot \|$ on $\mathbb{R}^2$, meaning that $d(x, y) = \|x - y\|$?

(b) Sketch the unit ball $B_1(0)$ in $(\mathbb{R}^2, d)$. Is it a convex set?

Solution

• (a) The symmetry and positivity of $d$ are immediate, so we just need to verify the triangle inequality.

• For any $a, b \geq 0$, we have

$$\left( \sqrt{a} + \sqrt{b} \right)^2 = a + 2\sqrt{ab} + b \geq a + b,$$

which shows that

$$\sqrt{a} + \sqrt{b} \geq \sqrt{a + b},$$

with equality if and only if $a = 0$ or $b = 0$.

• Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then, since $x \mapsto \sqrt{x}$ is an increasing function, the previous inequality implies that

$$d(x, y) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}$$

$$\leq \sqrt{|x_1 - z_1| + |z_1 - y_1|} + \sqrt{|x_2 - z_2| + |z_2 - y_2|}$$

$$\leq \sqrt{|x_1 - z_1| + |y_1 - z_1|} + \sqrt{|x_2 - z_2| + |y_2 - z_2|}$$

$$\leq d(x, z) + d(z, y).$$

• The metric is not derived from a norm on $\mathbb{R}^2$ since

$$d(\lambda x, \lambda y) = \sqrt{\lambda}d(x, y)$$

for $\lambda \in \mathbb{R}$, so it is not homogeneous of degree one.

• (b) The unit ball is shown in the figure. It is not convex. For example, if $1/2 \leq |a| < 1$ and

$$x = (a, 0), \ y = (0, a), \ z = \frac{1}{2}(x + y),$$
then \(d(x, 0) = d(y, 0) = \sqrt{a} < 1\) and \(d(z, 0) = \sqrt{2a} \geq 1\), so \(x, y \in B_1(0)\) but \(z \notin B_1(0)\).

**Remark.** The unit ball of a (real) normed space is always convex, since \(\|x\|, \|y\| < 1\) and \(0 \leq \lambda \leq 1\) implies that

\[
\|\lambda x + (1 - \lambda)y\| \leq \lambda\|x\| + (1 - \lambda)\|y\| < 1.
\]
Problem 5. Define the closure \( \bar{A} \) of a subset \( A \subset X \) of a metric space \( X \) by
\[
\bar{A} = \bigcap \{F \subset X : F \supset A \text{ and } F \text{ is closed}\}.
\]
Show that
\[
\bar{A} = \{x \in X : \text{there exists a sequence } (x_n) \text{ with } x_n \in A \text{ and } x_n \to x\}.
\]

Solution

• First, we show that \( x \in \bar{A} \) if and only if every neighborhood of \( x \) contains some point in \( A \). To do this, we prove the equivalent statement that \( x \notin \bar{A} \) if and only if some neighborhood of \( x \) is disjoint from \( A \).

• If \( x \notin \bar{A} \), then since \( \bar{A} \supseteq A \) is closed and \( \bar{A}^c \subset A^c \) is open, there is a neighborhood \( U_x \subset \bar{A}^c \) of \( x \) that is disjoint from \( A \).

• Conversely, if \( U_x \) is an open neighborhood of \( x \in X \) that is disjoint from \( A \), then \( F = U_x^c \) is a closed set with \( F \supset A \) and \( x \notin F \) so \( x \notin \bar{A} \).

• Let \( \hat{A} \) denote the sequential closure of \( A \):
\[
\hat{A} = \{x \in X : \text{there exists a sequence } (x_n) \text{ with } x_n \in A \text{ and } x_n \to x\}.
\]

• If \( x \notin \hat{A} \), then \( x \) has a neighborhood that is disjoint from \( \bar{A} \supset A \), so no sequence in \( A \) can converge to \( x \) and \( x \notin \hat{A} \). It follows that \( \bar{A} \supset \hat{A} \).

• If \( x \in \bar{A} \), then for every \( n \in \mathbb{N} \), there exists \( x_n \in B_{1/n}(x) \cap A \), so \( (x_n) \) is a sequence in \( A \) that converges to \( x \), and \( x \in \hat{A} \). It follows that \( \hat{A} \supset \bar{A} \), so \( \hat{A} = \bar{A} \).
Problem 6. Is the closure of the open ball

\[ B_r(x) = \{ y \in X : d(x, y) < r \} \]

in a metric space \((X, d)\) always equal to the closed ball

\[ \overline{B}_r(x) = \{ y \in X : d(x, y) \leq r \} \]?

Solution

• This is not true in general.

• For example, if \(X\) is a set with at least two elements and \(d : X \times X \to \mathbb{R}\) is the discrete metric,

\[ d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \]

then every subset of \(X\) is closed and \(B_1(x) = \{x\}, \overline{B_1(x)} = \{x\}\), but \(\overline{B}_1(x) = X\), so \(\overline{B}_1(x) \neq \overline{B}_1(x)\).
Problem 7. Let $X$ be the space of all real sequences of the form

$$x = (x_1, x_2, x_3, \ldots, x_N, 0, 0, \ldots)$$

for some $N \in \mathbb{N}$, where $x_n \in \mathbb{R}$, whose terms are zero from some point on. Define

$$\|x\| = \max_{n \in \mathbb{N}} |x_n|.$$ 

(a) Show that $(X, \| \cdot \|)$ is a normed linear space (with vector addition and scalar multiplication defined componentwise).

(b) Show that $X$ is not complete.

(c) Let $c_0$ denote the space of all real sequences $(x_n)$ such that $x_n \to 0$ as $n \to \infty$. Show that $(c_0, \| \cdot \|)$ is complete and $X$ is dense in $c_0$.

Solution.

• (a) It is immediate to verify that $X$ is a linear space under componentwise addition and scalar multiplication. (Note that a finite linear combination of sequences in $X$ also belongs to $X$.)

• The properties of a norm are straightforward to check. For example, if $x = (x_n)$ and $y = (y_n)$, then

$$\|x + y\| = \max_{n \in \mathbb{N}} |x_n + y_n|$$

$$\leq \max_{n \in \mathbb{N}} \{|x_n| + |y_n|\}$$

$$\leq \max_{n \in \mathbb{N}} |x_n| + \max_{n \in \mathbb{N}} |y_n|$$

$$\leq \|x\| + \|y\|.$$ 

• (b) Consider the sequence $(x^{(k)})$ in $X$ defined for $k \in \mathbb{N}$ by

$$x^{(k)} = (1, 1/2, 1/3, \ldots, 1/k, 0, 0, \ldots).$$

Then, for all $j > k$, we have

$$\|x^{(j)} - x^{(k)}\| = \frac{1}{k + 1},$$

so the sequence is Cauchy. However, if $x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$ is any point in $X$, then

$$\|x^{(k)} - x\| \geq \frac{1}{N + 1}$$

for all $k \geq N + 1$,

so the sequence $(x^{(k)})$ does not have a limit in $X$, and $X$ is not complete.
• (c) First, we show that $X$ is dense in $c_0$. If $x = (x_n) \in c_0$, then given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n| < \epsilon$ for all $n > N$. It follows that if

$$x^{(N)} = (x_1, \ldots, x_N, 0, 0, \ldots) \in X,$$

then $\|x - x^{(N)}\| < \epsilon$, so $X$ is a dense subspace of $c_0$.

• Next, we prove that $c_0$ is complete. Suppose that $(x^{(k)})$ is a Cauchy sequence in $c_0$, where

$$x^{(k)} = \left( x^{(k)}_n \right)_{n=1}^\infty.$$

Since

$$|x^{(k)}_n - x^{(\ell)}_n| \leq \|x^{(k)} - x^{(\ell)}\|,$$

the sequence $(x^{(k)}_n)_{k=1}^\infty$ is Cauchy in $\mathbb{R}$ for each $n \in \mathbb{N}$, so by the completeness of $\mathbb{R}$, there is $x_n \in \mathbb{R}$ such that

$$x^{(k)}_n \to x_n \quad \text{as} \quad k \to \infty.$$

• Let $x = (x_n)$ and let $\epsilon > 0$ be given. Then there exists $K_\epsilon \in \mathbb{N}$ such that

$$|x^{(k)}_n - x^{(\ell)}_n| < \epsilon \quad \text{for every} \quad n \in \mathbb{N} \quad \text{and all} \quad k, \ell \geq K_\epsilon.$$

Taking the limit of this inequality as $\ell \to \infty$, we get that

$$|x^{(k)}_n - x_n| \leq \epsilon \quad \text{for every} \quad n \in \mathbb{N} \quad \text{and all} \quad k \geq K_\epsilon.$$

It follows that that

$$\|x^{(k)} - x\| = \sup_{n \in \mathbb{N}} |x^{(k)}_n - x_n| \leq \epsilon \quad \text{for} \quad k \geq K_\epsilon,$$

which shows that $\|x^{(k)} - x\| \to 0$ as $k \to \infty$.

• Finally, we show that $x \in c_0$. Let $\epsilon > 0$ be given. Then there exists $k_\epsilon \in \mathbb{N}$ such that

$$\|x - x^{(k_\epsilon)}\| < \frac{\epsilon}{2},$$

and since $x^{(k_\epsilon)} \in c_0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$|x^{(k_\epsilon)}_n| < \frac{\epsilon}{2} \quad \text{for} \quad n > N_\epsilon.$$

It follows that

$$|x_n| \leq |x_n - x^{(k_\epsilon)}_n| + |x^{(k_\epsilon)}_n| < \epsilon \quad \text{for} \quad n > N_\epsilon,$$

which shows that $x \in c_0$ and $c_0$ is complete.