Problem 1. (a) Prove that a closed subset of a complete metric space is complete. (b) Prove that a closed subset of a compact metric space is compact. (c) Prove that a compact subset of a metric space is closed and bounded.

Solution

- (a) If $F \subset X$ is closed and $(x_n)$ is a Cauchy sequence in $F$, then $(x_n)$ is Cauchy in $X$ and $x_n \to x$ for some $x \in X$ since $X$ is complete. Then $x \in F$ since $F$ is closed, so $F$ is complete.

- (b) Suppose that $F \subset X$ where $F$ is closed and $X$ is compact. If $(x_n)$ is a sequence in $F$, then there is a subsequence $(x_{n_k})$ that converges to $x \in X$ since $X$ is compact. Then $x \in F$ since $F$ is closed, so $F$ is compact. Alternatively, If $\{G_\alpha \subset X : \alpha \in I\}$ is an open cover of $F$, then $\{G_\alpha : \alpha \in I\} \cup F^c$ is an open cover of $X$. Since $X$ is compact, there is a finite subcover of $X$ which also covers $F$, so $F$ is compact.

- (c) Let $K \subset X$ be compact. If $(x_n)$ is a convergent sequence in $K$ with limit $x \in X$, then every subsequence of $(x_n)$ converges to $x$. Since $K$ is compact, some subsequence of $(x_n)$ converges to a limit in $K$, so $x \in K$ and $K$ is closed.

- Suppose that $K$ is not bounded, and let $x_1 \in K$. Then for every $r > 0$ there exists $x \in K$ such that $d(x_1, x) \geq r$. Choose a sequence $(x_n)$ in $K$ as follows. Pick $x_2 \in K$ such that $d(x_1, x_2) \geq 1$. Given $\{x_1, x_2, \ldots, x_n\}$, pick $x_{n+1} \in K$ such that

$$d(x_1, x_{n+1}) \geq 1 + \max_{1 \leq k \leq n} d(x_1, x_k).$$

By the triangle inequality,

$$d(x_k, x_{n+1}) \geq d(x_1, x_{n+1}) - d(x_1, x_k) \geq 1 \quad \text{for every } 1 \leq k \leq n.$$

It follows that $d(x_m, x_n) \geq 1$ for every $m \neq n$, so $(x_n)$ has no Cauchy subsequences, and therefore no convergent subsequences, so $K$ is not compact.
Problem 2. Let $A$ be a subset of a metric space $X$ with closure $\bar{A}$. Define the interior $A^\circ$ and boundary $\partial A$ of $A$ by

$$A^\circ = \bigcup \{ G \subset A : G \text{ is open} \}, \quad \partial A = \bar{A} \setminus A^\circ.$$ 

(a) Why is $A^\circ$ open and $\partial A$ closed?
(b) Prove that $X \setminus \bar{A} = (X \setminus A)^\circ$.
(c) Prove that $A$ is closed if and only if $\partial A \subset A$, and $A$ is open if and only if $\partial A \subset A^c$.
(d) If $A$ is open, does it follow that $(\bar{A})^\circ = A$?

Solution

- (a) A union of open sets is open so $A^\circ$ is open, and an intersection of closed sets is closed so $\bar{A}$ and $\partial A = \bar{A} \cap (A^\circ)^c$ are closed.

- (b) Note that $x \in A^\circ$ if and only $B_\epsilon(x) \subset A$ for some $\epsilon > 0$. If $x \in \bar{A}^c$, then $B_\epsilon(x) \subset \bar{A}^c$ for some $\epsilon > 0$ since $\bar{A}^c$ is open. Since $\bar{A} \supset A$, we have $\bar{A}^c \subset A^c$, so $B_\epsilon(x) \subset A^c$, meaning that $x \in (A^c)^\circ$. It follows that $\bar{A}^c \subset (A^c)^\circ$. For the reverse inclusion, note that if $x \in (A^c)^\circ$, then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset A^c$, so $x$ is not the limit of any sequence in $A$, meaning that $x \in \bar{A}^c$. It follows that $(A^c)^\circ \subset \bar{A}^c$, so $(A^c)^\circ = \bar{A}^c$.

- (c) If $A$ is closed, then $\bar{A} = A$ so $\partial A = A \cap (A^\circ)^c \subset A$. For the converse, note that since $A^\circ \subset A \subset \bar{A}$, we have

$$\bar{A} = (\bar{A} \cap A^\circ) \cup (\bar{A} \cap (A^\circ)^c) = A^\circ \cup \partial A.$$ 

If $\partial A \subset A$, then it follows that $\bar{A} \subset A$, so $A = \bar{A}$, meaning that $A$ is closed.

- (d) This is not true in general. For example, define $A \subset \mathbb{R}^2$ by

$$A = \{(x, y) : x^2 + y^2 < 1 \} \setminus \{(x, 0) : 0 \leq x < 1 \}.$$ 

Then $A$ is open, but $(\bar{A})^\circ = \{(x, y) : x^2 + y^2 < 1 \} \neq A$.
**Problem 3.** Let $X$ be a metric space with a dense subset $A \subset X$ such that every Cauchy sequence in $A$ converges in $X$. Prove that $X$ is complete.

**Solution**

- Let $(x_n)$ be a Cauchy sequence in $X$. Since $A$ is dense in $X$, we can choose a sequence $(a_n)$ in $A$ such that $d(x_n, a_n) \to 0$ as $n \to \infty$. (For example, choose $a_n \in A$ such that $d(x_n, a_n) < 1/n$.)

- Given any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/3$ and $d(x_m, x_n) < \epsilon/3$ for all $m, n > M$, so

$$d(a_m, a_n) \leq d(a_m, x_m) + d(x_m, x_n) + d(x_n, a_n) < \epsilon,$$

which shows that $(a_n)$ is a Cauchy sequence in $A$. It follows that $(a_n)$ converges to some $x \in X$.

- Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/2$ and $d(a_n, x) < \epsilon/2$ for all $n > N$, so

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \epsilon,$$

which shows that $(x_n)$ converges to $x$ and proves that $X$ is complete.
Problem 4. Let $d : X \times X \to \mathbb{R}$ be the discrete metric on a set $X$,

$$d(x, y) = \begin{cases} 
1 & \text{if } x \neq y, \\
0 & \text{if } x = y.
\end{cases}$$

What are the compact subsets of the metric space $(X, d)$?

Solution

- A subset of $X$ is compact if and only if it is finite.
- Every finite set is compact. If $F = \{x_1, x_2, \ldots, x_n\} \subset X$ and
  $$\{G_\alpha \subset X : \alpha \in I\}$$

  is an open cover of $F$, then $x_k \in G_{\alpha_k}$ for some $\alpha_k \in I$, so
  $$\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$$

  is a finite subcover of $F$. Alternatively, every sequence in $F$ has a constant subsequence, which converges to a point in $F$.

- Conversely, if $F \subset X$ is infinite and $G_x = \{x\}$, then $\{G_x : x \in F\}$

  is an open cover of $F$ with no finite subcover. Alternatively, if $F$ is infinite, then there is a sequence $(x_n)$ in $F$ with $x_m \neq x_n$ for all $m \neq n$,
  so $d(x_m, x_n) = 1$ for $m \neq n$, and $(x_n)$ has no Cauchy or convergent subsequences.

Remark. A rough heuristic is that compact sets have many properties in common with finite sets. For example, finite sets have the finite intersection property.
Problem 5. Let $c_0$ be the Banach space of real sequences $(x_n)$ such that $x_n \to 0$ as $n \to \infty$ with the sup-norm $\|(x_n)\| = \sup_{n \in \mathbb{N}} |x_n|$. Is the closed unit ball

$$B = \{(x_n) \in c_0 : \|(x_n)\| \leq 1\}$$

compact?

Solution

• The closed unit ball in $c_0$ is not compact.

• For example, let

$$e_k = (\delta_{nk})_{n=1}^\infty \quad \delta_{nk} = \begin{cases} 
1 & \text{if } n = k \\
0 & \text{if } n \neq k
\end{cases}$$

denote the sequence whose $k$th term is one and whose other terms are zero. Then $e_k \in c_0$ since $\lim_{n \to \infty} \delta_{nk} = 0$, and $e_k$ belongs to the closed unit ball in $c_0$ since $\|e_k\| = 1$. However, $\|e_j - e_k\| = 1$ for every $j \neq k$, so $(e_k)_{k=1}^\infty$ has no Cauchy or convergent subsequences in $c_0$.

Remark. A similar argument using the Riesz lemma shows that the closed unit ball in any infinite-dimensional normed space is not compact in the norm topology.
Problem 6. A metric (or topological) space $X$ is disconnected if there are non-empty open sets $U, V \subset X$ such that $X = U \cup V$ and $U \cap V = \emptyset$. A space is connected if it is not disconnected. A space $X$ is totally disconnected if its only non-empty connected subsets are the singleton sets $\{x\}$ with $x \in X$.

(a) Show that the interval $[0, 1]$ is connected (in its standard metric topology).

(b) Show that the set $\mathbb{Q}$ of rational numbers is totally disconnected.

Solution

• (a) Suppose for contradiction that $[0, 1] = U \cup V$ where $U, V$ are nonempty, disjoint open sets in $[0, 1]$. We assume that $0 \in U$ without loss of generality.

• Let $a = \sup \{x \in [0, 1] : [0, x) \subset U\}$. Since $0 \in U$ and $U$ is open, we have $[0, \epsilon) \subset U$ for some $\epsilon > 0$, so $0 < a \leq 1$. If $0 < b < a$, then $[0, b) \subset U$ since, by the definition of the supremum, there exists $b < c < a$ such that $[0, c) \subset U$. (It also follows that $[0, a) = \bigcup_{0 < b < a} [0, b) \subset U$, so the supremum is attained and, in fact, $\{x \in [0, 1] : [0, x) \subset U\} = [0, a]$.)

• If $a \in U$, then $(a-\epsilon, a+\epsilon) \subset U$ for some $\epsilon > 0$, but then $[0, a+\epsilon) \subset U$, contradicting the definition of $a$. On the other hand, if $a \in V$, then $(a-\epsilon, a] \subset V$ for some $0 < \epsilon < a$, but then $[0, b) \not\subset U$ for $a-\epsilon < b \leq a$, also contradicting the definition of $a$.

• (b) Let $A \subset \mathbb{Q}$ be any subset of the rational numbers with at least two elements. Choose $x, y \in A$ with $x \neq y$. The irrational numbers are dense in $\mathbb{R}$, so there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$. Let

$$U = (-\infty, z) \cap A, \quad V = (z, \infty) \cap A.$$ 

Then $U, V$ are open sets in the relative topology on $A$. Moreover, $x \in U$, $y \in V$ so $U, V$ are nonempty, and $U \cap V = \emptyset$, $U \cup V = A$. It follows that $\mathbb{Q}$ is totally disconnected.
Problem 7. Let
\[ T = \{ G \subset \mathbb{R} : G^c \text{ is countable or } G^c = \mathbb{R} \}. \]

(a) Show that \( T \) is a topology on \( \mathbb{R} \).
(b) Let \( I = (0, 1) \) with closure \( \bar{I} = \bigcap \{ F \supset I : F \text{ is closed} \} \) in this topology. Show that \( \bar{I} = \mathbb{R} \).
(c) Is there a sequence \((x_n)\) such that \( x_n \in I \) and \( x_n \to 2 \) in this topology?
(d) What part of your proof in Problem 5 of Set 1 fails in this example?

Solution

- (a) The collection of closed sets \( C = \{ F \subset \mathbb{R} : F^c \in T \} \) in this topology is given by \( C = \{ F \subset \mathbb{R} : F \text{ is countable or } F = \mathbb{R} \} \).
- The collection \( C \) satisfies the axioms for closed sets in a topological space: (1) \( \emptyset, \mathbb{R} \in C \). (2) The intersection of closed sets is closed, since either every set is \( \mathbb{R} \) and the intersection is \( \mathbb{R} \), or at least one set is countable and the intersection in countable, since any subset of a countable set is countable. (3) A finite union of closed sets is closed, since a finite (or countable) union of countable sets is countable. It follows that \( T \) is a topology on \( \mathbb{R} \) (called the co-countable topology).
- (b) If \( F \supset I \) is closed, then \( F \) is uncountable, since \((0, 1)\) is uncountable, so \( F = \mathbb{R} \) and \( \bar{I} = \mathbb{R} \).
- (c) Let \((x_n)_{n=1}^{\infty}\) be a sequence in \((0, 1)\). Then \( U = \mathbb{R} \setminus \{ x_n : n \in \mathbb{N} \} \) is an open neighborhood of \( 2 \), but \( x_n \notin U \) for any \( n \in \mathbb{N} \), so \((x_n)\) does not converge to \( 2 \). A similar argument applies to any \( x \notin (0, 1) \), so the sequential closure of \( I \) is \( \bar{I} = (0, 1) \).
- (d) If \( X \) is a topological space, then a neighborhood base of \( x \in X \) is a collection \( \{ U_\alpha : \alpha \in A \} \) of neighborhoods of \( x \) such that for every neighborhood \( U \) of \( x \) there exists \( \alpha \in A \) with \( U_\alpha \subset U \). Then \( x_n \to x \) if and only if for every \( \alpha \in A \) there exists \( N \in \mathbb{N} \) such that \( x_n \in U_\alpha \) for all \( n > N \). The proof that the sequential closure is equal to the closure fails for the co-countable topology on \( \mathbb{R} \) because \( x \in \mathbb{R} \) does not have a countable neighborhood base. On the other hand, if \( X \) is a metric space, then \( \{ B_1/n(x) : n \in \mathbb{N} \} \) is a countable neighborhood base of any \( x \in X \).
Problem 8. Let $S$ be the set of sequences whose terms are 0 or 1:

$$S = \{(s_k)_{k=1}^\infty : s_k = 0 \text{ or } s_k = 1\}.$$  

(a) Use a diagonal argument to show that $S$ is uncountable.

(b) Show that $S$ has the same cardinality as the power set $\mathcal{P}(\mathbb{N})$ of the natural numbers.

Solution

- (a) Let $f : \mathbb{N} \to S$. Suppose that $f(n) = s_n$ where $s_n = (s_{kn})_{k=1}^\infty$, and define $t = (t_k)_{k=1}^\infty \in S$ by

  $$t_k = \begin{cases} 
    1 & \text{if } s_{kk} = 0, \\
    0 & \text{if } s_{kk} = 1.
  \end{cases}$$

  Then $t \neq s_k$ for every $k \in \mathbb{N}$, since the two sequences have different $k$th terms. It follows that there is no map from $\mathbb{N}$ onto $S$, so $S$ is uncountably infinite.

- For $A \subset \mathbb{N}$ define the characteristic function $\chi_A : \mathbb{N} \to \{0, 1\}$ of $A$ by

  $$\chi_A(k) = \begin{cases} 
    1 & \text{if } k \in A, \\
    0 & \text{if } k \notin A.
  \end{cases}$$

  Then the map $f : \mathcal{P}(\mathbb{N}) \to S$ defined by $f(A) = (\chi_A(k))_{k=1}^\infty$ is one-to-one and onto, so $\mathcal{P}(\mathbb{N})$ and $S$ have the same cardinality.